16 THE AXIOMATIC AND STOCHASTIC APPROACHES TO INDEX NUMBER THEORY

Introduction

16.1 As was seen in Chapter 15, it is useful to be able to evaluate various index number formulae that have been proposed in terms of their properties. If a formula turns out to have rather undesirable properties, this casts doubts on its suitability as an index that could be used by a statistical agency as a target index. Looking at the mathematical properties of index number formulae leads to the *test* or *axiomatic approach to index number theory*. In this approach, desirable properties for an index number formula are proposed, and it is then attempted to determine whether any formula is consistent with these properties or tests. An ideal outcome is the situation where the proposed tests are both desirable and completely determine the functional form for the formula.

16.2 The axiomatic approach to index number theory is not completely straightforward, since choices have to be made in two dimensions:

- The index number framework must be determined.
- Once the framework has been decided upon, it must be decided what tests or properties should be imposed on the index number.

The second point is straightforward: different price statisticians may have different ideas about which tests are important, and alternative sets of axioms can lead to alternative "best" index number functional forms. This point must be kept in mind while reading this chapter, since there is no universal agreement on what the "best" set of "reasonable" axioms is. Hence the axiomatic approach can lead to more than one best index number formula.

16.3 The first point about choices listed above requires further discussion. In the previous chapter, for the most part, the focus was on *bilateral index number theory*; i.e., it was assumed that prices and quantities for the same *n* commodities were given for two periods and the object of the index number formula was to compare the overall level of prices in one period with the other period. In this framework, both sets of price and quantity vectors were regarded as variables which could be independently varied so that, for example, variations in the prices of one period did not affect the prices of the other period or the quantities in either period. The emphasis was on comparing the overall cost of a fixed basket of quantities in the two periods or taking averages of such fixed basket indices. This is an example of an index number framework.

16.4 However, other index number frameworks are possible. For example, instead of decomposing a value ratio into a term that represents price change between the two periods times another term that represents quantity change, an attempt could be made to decompose a value aggregate for one period into a single number that represents the price level in the period times another number that represents the quantity level in the period. In the first variant of this approach, the price index number is supposed to be a function of the n commodity prices pertaining to that aggregate in the period under consideration, while the quantity index number is supposed to be a function of the n commodity quantities pertaining to the aggregate in the period. The resulting price index function was called an *absolute index number* by Frisch (1930, p. 397), a *price level* by Eichhorn (1978, p. 141) and a *unilateral price index* by Anderson, Jones and Nesmith (1997, p. 75). In a second variant of this approach, the price and quantity functions are allowed to depend on both the price and quantity vectors pertaining to the period under consideration.[1] These two variants of unilateral index number theory will be considered in paragraphs 16.11 to 16.29.[2]

16.5 The remaining approaches in this chapter are largely bilateral approaches; i.e., the prices and quantities in an aggregate are compared for two periods. In paragraphs 16.30 to 16.73 and 16.94 to 16.129, the value ratio decomposition approach is taken.[3] In paragraphs 16.30 to 16.73, the bilateral price and quantity indices, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, are regarded as functions of the price vectors pertaining to the two periods, p^0 and p^1, and the two quantity vectors, q^0 and q^1. Not only do the axioms or tests that are placed on the price index $P(p^0,p^1,q^0,q^1)$ reflect "reasonable" price index properties, but some tests have their origin as "reasonable" tests on the quantity index $Q(p^0,p^1,q^0,q^1)$. The approach in paragraphs 16.30 to 16.73 simultaneously determines the "best" price and quantity indices.

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[1] Eichhorn (1978, p. 144) and Diewert (1993d, p. 9) considered this approach.

[2] In these unilateral index number approaches, the price and quantity vectors are allowed to vary independently. In yet another index number framework, prices are allowed to vary freely but quantities are regarded as functions of the prices. This leads to the *economic approach to index number theory*, which is considered briefly in Appendix 15.4 of Chapter 15, and in more depth in Chapters 17 and 18.

[3] Recall paragraphs 15.7 to 15.17 of Chapter 15 for an explanation of this approach.

16.6 In paragraphs 16.74 to 16.93, attention is shifted to the *price ratios* for the *n* commodities between periods 0 and 1, $r_i \equiv p_i^1/p_i^0$ for $i = 1,\ldots,n$. In the *unweighted stochastic approach to index number theory*, the price index is regarded as an evenly weighted average of the *n* price relatives or ratios, r_i. Carli (1764) and Jevons (1863; 1865) were the earlier pioneers in this approach to index number theory, with Carli using the arithmetic average of the price relatives and Jevons endorsing the geometric average (but also considering the harmonic average). This approach to index number theory will be covered in paragraphs 16.74 to 16.79. This approach is consistent with a statistical approach that regards each price ratio r_i as a random variable with mean equal to the underlying price index.

16.7 A major problem with the unweighted average of price relatives approach to index number theory is that this approach does not take into account the economic importance of the individual commodities in the aggregate. Young (1812) did advocate some form of rough weighting of the price relatives according to their relative value over the period being considered, but the precise form of the required value weighting was not indicated.[4] It was Walsh (1901, pp. 83-121; 1921a, pp. 81-90), however, who stressed the importance of weighting the individual price ratios, where the weights are functions of the associated values for the commodities in each period and each period is to be treated symmetrically in the resulting formula:

> What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [price ratios] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered (Walsh (1901, p. 104)).
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> Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period and there is a second period which is compared with it. Price variations[5] have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the second period are apt to be different from their weights at the first period. Which weights, then, are the

[4] Walsh (1901, p. 84) refers to Young's contributions as follows:

> Still, although few of the practical investigators have actually employed anything but even weighting, they have almost always recognized the theoretical need of allowing for the relative importance of the different classes ever since this need was first pointed out, near the commencement of the century just ended, by Arthur Young. … Arthur Young advised simply that the classes should be weighted according to their importance.

[5] A price variation is a price ratio or price relative in Walsh's terminology.

> right ones – those of the first period or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights of the two periods (Walsh (1921a, p. 90)).

16.8 Thus Walsh was the first to examine in some detail the rather intricate problems[6] involved in deciding how to weight the price relatives pertaining to an aggregate, taking into account the economic importance of the commodities in the two periods being considered. Note that the type of index number formula that Walsh was considering was of the form $P(r,v^0,v^1)$, where r is the vector of price relatives which has ith component $r_i = p_i^1/p_i^0$ and v^t is the period t value vector which has ith component $v_i^t = p_i^t q_i^t$ for $t = 0,1$. His suggested solution to this weighting problem was not completely satisfactory but he did at least suggest a very useful framework for a price index, as a value-weighted average of the n price relatives. The first satisfactory solution to the weighting problem was obtained by Theil (1967, pp. 136-137) and his solution is explained in paragraphs 16.79 to 16.93.

16.9 It can be seen that one of Walsh's approaches to index number theory[7] was an attempt to determine the "best" weighted average of the price relatives, r_i. This is equivalent to using an axiomatic approach to try to determine the "best" index of the form $P(r,v^0,v^1)$. This approach is considered in paragraphs 16.94 to 16.129.[8]

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[6] Walsh (1901, pp. 104-105) realized that it would not do to simply take the arithmetic average of the values in the two periods, $[v_i^0 + v_i^1]/2$, as the "correct" weight for the ith price relative r_i since, in a period of rapid inflation, this would give too much importance to the period that had the highest prices and he wanted to treat each period symmetrically:

> But such an operation is manifestly wrong. In the first place, the sizes of the classes at each period are reckoned in the money of the period, and if it happens that the exchange value of money has fallen, or prices in general have risen, greater influence upon the result would be given to the weighting of the second period; or if prices in general have fallen, greater influence would be given to the weighting of the second period. Or in a comparison between two countries greater influence would be given to the weighting of the country with the higher level of prices. But it is plain that *the one period, or the one country, is as important, in our comparison between them, as the other, and the weighting in the averaging of their weights should really be even*.

However, Walsh was unable to come up with Theil's (1967) solution to the weighting problem, which was to use the average expenditure share $[s_i^0 + s_i^1]/2$, as the "correct" weight for the ith price relative in the context of using a weighted geometric mean of the price relatives.

[7] Walsh also considered basket-type approaches to index number theory, as was seen in Chapter 15.

[8] In paragraphs 16.94 to 16.129, rather than starting with indices of the form $P(r,v^0,v^1)$, indices of the form $P(p^0,p^1,v^0,v^1)$ are considered. However, if the test of invariance to changes in the units of measurement is imposed on this index, it is equivalent to studying indices of the form $P(r,v^0,v^1)$. Vartia (1976) also used a variation of this approach to index number theory.

16.10 The Young and Lowe indices, discussed in Chapter 15, do not fit precisely into the bilateral framework since the value or quantity weights used in these indices do not necessarily correspond to the values or quantities that pertain to either of the periods that correspond to the price vectors p^0 and p^1. The axiomatic properties of these two indices with respect to their price variables are studied in paragraphs 16.130 to 16.134.

The levels approach to index number theory

An axiomatic approach to unilateral price indices

16.11 Denote the price and quantity of commodity *i* in period *t* by p_i^t and q_i^t respectively for $i = 1,2,\ldots,n$ and $t = 0,1,\ldots,T$. The variable q_i^t is interpreted as the total amount of commodity *i* transacted within period *t*. In order to conserve the value of transactions, it is necessary that p_i^t be defined as a unit value; i.e., p_i^t must be equal to the value of transactions in commodity *i* for period *t* divided by the total quantity transacted, q_i^t. In principle, the period of time should be chosen so that variations in commodity prices within a period are very small compared to their variations between periods.[9] For $t = 0,1,\ldots,T$, and $i = 1,\ldots,n$, define the value of transactions in commodity *i* as $v_i^t \equiv p_i^t q_i^t$ and define the *total value of transactions in period t* as:

$$V^t \equiv \sum_{i=1}^{n} v_i^t = \sum_{i=1}^{n} p_i^t q_i^t \qquad t = 0,1,...,T. \qquad (16.1)$$

[9] This treatment of prices as unit values over time follows Walsh (1901, p. 96; 1921a, p. 88) and Fisher (1922, p. 318). Fisher and Hicks both had the idea that the length of the period should be short enough so that variations in price within the period could be ignored, as the following quotations indicate:

> Throughout this book "the price" of any commodity or "the quantity" of it for any one year was assumed given. But what is such a price or quantity? Sometimes it is a single quotation for January 1 or July 1, but usually it is an average of several quotations scattered throughout the year. The question arises: On what principle should this average be constructed? The *practical* answer is *any* kind of average since, ordinarily, the variation during a year, so far, at least, as prices are concerned, are too little to make any perceptible difference in the result, whatever kind of average is used. Otherwise, there would be ground for subdividing the year into quarters or months until we reach a small enough period to be considered practically a point. The quantities sold will, of course, vary widely. What is needed is their sum for the year (which, of course, is the same thing as the simple arithmetic average of the per annum rates for the separate months or other subdivisions). In short, the simple arithmetic average, both of prices and of quantities, may be used. Or, if it is worth while to put any finer point on it, we may take the weighted arithmetic average for the prices, the weights being the quantities sold (Fisher (1922, p. 318)).
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> I shall define a week as that period of time during which variations in prices can be neglected. For theoretical purposes this means that prices will be supposed to change, not continuously, but at short intervals. The calendar length of the week is of course quite arbitrary; by taking it to be very short, our theoretical scheme can be fitted as closely as we like to that ceaseless oscillation which is a characteristic of prices in certain markets (Hicks (1946, p. 122)).

16.12 Using the above notation, the following *levels version of the index number problem* is defined as follows: for $t = 0,1,\ldots,T$, find scalar numbers P^t and Q^t such that

$$V^t = P^t Q^t \qquad t = 0,1,\ldots,T. \qquad (16.2)$$

The number P^t is interpreted as an aggregate period t price level, while the number Q^t is interpreted as an aggregate period t quantity level. The aggregate price level P^t is allowed to be a function of the period t price vector, p^t, while the aggregate period t quantity level Q^t is allowed to be a function of the period t quantity vector, q^t; hence:

$$P^t = c(p^t) \text{ and } Q^t = f(q^t) \qquad t = 0,1,\ldots,T. \qquad (16.3)$$

16.13 The functions c and f are to be determined somehow. Note that equation (16.3) requires that the functional forms for the price aggregation function c and for the quantity aggregation function f be independent of time. This is a reasonable requirement since there is no reason to change the method of aggregation as time changes.

16.14 Substituting equations (16.3) and (16.2) into equation (16.1) and dropping the superscripts t means that c and f must satisfy the following functional equation for all strictly positive price and quantity vectors:

$$c(p)f(q) = \sum_{i=1}^{n} p_i q_i \qquad \text{for all } p_i > 0 \text{ and for all } q_i > 0. \qquad (16.4)$$

16.15 It is natural to assume that the functions $c(p)$ and $f(q)$ are positive if all prices and quantities are positive:

$$c(p_1,\ldots,p_n) > 0;\; f(q_1,\ldots,q_n) > 0 \quad \text{if all } p_i > 0 \text{ and all } q_i > 0. \qquad (16.5)$$

16.16 Let 1_n denote an n-dimensional vector of ones. Then (16.5) implies that when $p = 1_n$, $c(1_n)$ is a positive number, a for example, and when $q = 1_n$, then $f(1_n)$ is also a positive number, b for example; i.e., (16.5) implies that c and f satisfy:

$$c(1_n) = a > 0;\; f(1_n) = b > 0. \qquad (16.6)$$

16.17 Let $p = 1_n$ and substitute the first equation in (16.6) into equation (16.4) in order to obtain the following equation:

$$f(q) = \sum_{i=1}^{n} \frac{q_i}{a} \qquad \text{for all } q_i > 0. \qquad (16.7)$$

16.18 Now let $q = 1_n$ and substitute the second equation in (16.6) into equation (16.4) in order to obtain the following equation:

$$c(p) = \sum_{i=1}^{n} \frac{p_i}{b} \qquad \text{for all } p_i > 0. \qquad (16.8)$$

16.19 Finally substitute equations (16.7) and (16.8) into the left-hand side of equation (16.4) to obtain the following equation:

$$\left(\sum_{i=1}^{n} \frac{p_i}{b}\right)\left(\sum_{i=1}^{n} \frac{q_i}{a}\right) = \sum_{i=1}^{n} p_i q_i \qquad \text{for all } p_i > 0 \text{ and for all } q_i > 0. \qquad (16.9)$$

If *n* is greater than one, it is obvious that equation (16.9) cannot be satisfied for all strictly positive *p* and *q* vectors. Thus if the number of commodities *n* exceeds one, then there do not exist any functions *c* and *f* that satisfy equations (16.4) and (16.5).[10]

16.20 Thus this levels test approach to index number theory comes to an abrupt halt; it is fruitless to look for price and quantity level functions, $P^t = c(p^t)$ and $Q^t = f(q^t)$, that satisfy equations (16.2) or (16.4) and also satisfy the very reasonable positivity requirements (16.5).

16.21 Note that the levels price index function, $c(p^t)$, did not depend on the corresponding quantity vector q^t and the levels quantity index function, $f(q^t)$, did not depend on the price vector p^t. Perhaps this is the reason for the rather negative result obtained above. Hence, in the next section, the price and quantity functions are allowed to be functions of both p^t and q^t.

A second axiomatic approach to unilateral price indices

16.22 In this section, the goal is to find functions of 2*n* variables, $c(p,q)$ and $f(p,q)$ such that the following counterpart to equation (16.4) holds:

$$c(p,q)\, f(p,q) = \sum_{i=1}^{n} p_i q_i \qquad \text{for all } p_i > 0 \text{ and for all } q_i > 0. \qquad (16.10)$$

16.23 Again, it is natural to assume that the functions $c(p,q)$ and $f(p,q)$ are positive if all prices and quantities are positive:

[10] Eichhorn (1978, p. 144) established this result.

$$c(p_1,\ldots,p_n;q_1,\ldots,q_n) > 0;\; f(p_1,\ldots,p_n;q_1,\ldots,q_n) > 0 \text{ if all } p_i > 0 \text{ and all } q_i > 0. \quad (16.11)$$

16.24 The present framework does not distinguish between the functions *c* and *f,* so it is necessary to require that these functions satisfy some "reasonable" properties. The first property imposed on *c* is that this function be homogeneous of degree one in its price components:

$$c(\lambda p,q) = \lambda\, c(p,q) \qquad \text{for all } \lambda > 0. \qquad (16.12)$$

Thus, if all prices are multiplied by the positive number λ, then the resulting price index is λ times the initial price index. A similar linear homogeneity property is imposed on the quantity index *f*; i.e., *f* is to be homogeneous of degree one in its quantity components:

$$f(p,\lambda q) = \lambda\, f(p,q) \qquad \text{for all } \lambda > 0. \qquad (16.13)$$

16.25 Note that properties (16.10), (16.11) and (16.13) imply that the price index $c(p,q)$ has the following homogeneity property with respect to the components of q:

$$\begin{aligned} c(p,\lambda q) &= \sum_{i=1}^{n} \frac{p_i \lambda q_i}{f(p,\lambda q)} && \text{where } \lambda > 0 \\ &= \sum_{i=1}^{n} \frac{p_i \lambda q_i}{\lambda f(p,q)} && \text{using (16.13)} \\ &= \sum_{i=1}^{n} \frac{p_i q_i}{f(p,q)} \\ &= c(p,q) && \text{using (16.10) and (16.11).} \end{aligned} \qquad (16.14)$$

Thus $c(p,q)$ is homogeneous of degree 0 in its q components.

16.26 A final property that is imposed on the levels price index $c(p,q)$ is the following one. Let the positive numbers d_i be given. Then it is asked that the price index be invariant to changes in the units of measurement for the *n* commodities so that the function $c(p,q)$ has the following property:

$$c(d_1 p_1,\ldots,d_n p_n; q_1/d_1,\ldots,q_n/d_n) = c(p_1,\ldots,p_n;q_1,\ldots,q_n). \qquad (16.15)$$

16.27 It is now possible to show that properties (16.10), (16.11), (16.12), (16.14) and (16.15) on the price levels function $c(p,q)$ are inconsistent; i. e., there does not exist a function of $2n$ variables $c(p,q)$ that satisfies these very reasonable properties.[11]

16.28 To see why this is so, apply the equation (16.15), setting $d_i = q_i$ for each i, to obtain the following equation:

$$c(p_1,...,p_n;q_1,...,q_n) = c(p_1q_1,...,p_nq_n;1,...,1). \qquad (16.16)$$

If $c(p,q)$ satisfies the linear homogeneity property (16.12) so that $c(\lambda p,q) = \lambda c(p,q)$, then equation (16.16) implies that $c(p,q)$ is also linearly homogeneous in q so that $c(p,\lambda q) = \lambda c(p,q)$. But this last equation contradicts equation (16.14), which establishes the impossibility result.

16.29 The rather negative results obtained in paragraphs 16.13 to 16.21 indicate that it is fruitless to pursue the axiomatic approach to the determination of price and quantity levels, where both the price and quantity vector are regarded as independent variables.[12] Hence, in the following sections of this chapter, the axiomatic approach to the determination of a *bilateral price index* of the form $P(p^0,p^1,q^0,q^1)$ will be pursued.

The first axiomatic approach to bilateral price indices

Bilateral indices and some early tests

16.30 In this section, the strategy will be to assume that the bilateral price index formula, $P(p^0,p^1,q^0,q^1)$, satisfies a sufficient number of "reasonable" tests or properties so that the functional form for P is determined.[13] The word "bilateral"[14] refers to the assumption that the function P depends only on the data pertaining to the two situations or periods being compared; i.e., P is regarded as a function of the two sets of price and quantity vectors,

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[11] This proposition is due to Diewert (1993d, p. 9), but his proof is an adaptation of a closely related result due to Eichhorn(1978, pp. 144-145).

[12] Recall that in the economic approach, the price vector p is allowed to vary independently, but the corresponding quantity vector q is regarded as being determined by p.

[13] Much of the material in this section is drawn from sections 2 and 3 of Diewert (1992a). For more recent surveys of the axiomatic approach see Balk (1995) and Auer (2001).

[14] Multilateral index number theory refers to the case where there are more than two situations whose prices and quantities need to be aggregated.

p^0,p^1,q^0,q^1, that are to be aggregated into a single number that summarizes the overall change in the n price ratios, $p_1^1/p_1^0,\ldots, p_n^1/p_n^0$.

16.31 In this section, the value ratio decomposition approach to index number theory will be taken; i.e., along with the price index $P(p^0,p^1,q^0,q^1)$, there is a companion quantity index $Q(p^0,p^1,q^0,q^1)$ such that the product of these two indices equals the value ratio between the two periods.[15] Thus, throughout this section, it is assumed that P and Q satisfy the following *product test*:

$$V^1/V^0 = P(p^0,p^1,q^0,q^1)\, Q(p^0,p^1,q^0,q^1). \qquad (16.17)$$

The period t values, V^t, for t = 0,1 are defined by equation (16.1). As soon as the functional form for the price index P is determined, then equation (16.17) can be used to determine the functional form for the quantity index Q. A further advantage of assuming that the product test holds is that, if a reasonable test is imposed on the quantity index Q, then equation (16.17) can be used to translate this test on the quantity index into a corresponding test on the price index P.[16]

16.32 If $n = 1$, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0 , the single price ratio, and a natural candidate for Q is q_1^1/q_1^0 , the single quantity ratio. When the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one good price index formula, p_1^1/p_1^0. Below, some 20 tests are listed that turn out to characterize the Fisher ideal price index.

16.33 It will be assumed that every component of each price and quantity vector is positive; i.e., $p^t >> 0_n$ and $q^t >> 0_n$ [17] for t = 0,1. If it is desired to set $q^0 = q^1$, the common quantity vector is denoted by q; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

[15] See paragraphs 15.7 to 15.25 of Chapter 15 for more on this approach, which was initially due to Fisher (1911, p. 403; 1922).

[16] This observation was first made by Fisher (1911, pp. 400-406), and the idea was pursued by Vogt (1980) and Diewert (1992a).

[17] The notation $q >> 0_n$ means that each component of the vector q is positive; $q \geq 0_n$ means each component of q is non-negative and $q > 0_n$ means $q \geq 0_n$ and $q \neq 0_n$.

16.34 The first two tests, denoted T1 and T2, are not very controversial, so they will not be discussed in detail.

T1: *Positivity*:[18] $P(p^0,p^1,q^0,q^1) > 0$.

T2: *Continuity*:[19] $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

16.35 The next two tests, T3 and T4, are somewhat more controversial.

T3: *Identity or constant prices test*:[20] $P(p,p,q^0,q^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial aspect of this test is that the two quantity vectors are allowed to be different in the test.[21]

T4: *Fixed basket or constant quantities test*:[22] $P(p^0, p^1, q, q) = \dfrac{\sum_{i=1}^{n} p_i^1 q_i}{\sum_{i=1}^{n} p_i^0 q_i}$.

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^{n} p_i^1 q_i$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^{n} p_i^0 q_i$.

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[18] Eichhorn and Voeller (1976, p. 23) suggested this test.

[19] Fisher (1922, pp. 207-215) informally suggested the essence of this test.

[20] Laspeyres (1871, p. 308), Walsh (1901, p. 308) and Eichhorn and Voeller (1976, p. 24) have all suggested this test. Laspeyres came up with this test or property to discredit the ratio of unit values index of Drobisch (1871a), which does not satisfy this test. This test is also a special case of Fisher's (1911, pp. 409-410) price proportionality test.

[21] Usually, economists assume that, given a price vector *p*, the corresponding quantity vector *q* is uniquely determined. Here, the same price vector is used but the corresponding quantity vectors are allowed to be different.

[22] The origins of this test go back at least 200 years to the Massachusetts legislature, which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution; see Willard Fisher (1913). Other researchers who have suggested the test over the years include: Lowe (1823, Appendix, p. 95), Scrope (1833, p. 406), Jevons (1865), Sidgwick (1883, pp. 67-68), Edgeworth (1925, p. 215) originally published in 1887, Marshall (1887, p. 363), Pierson (1895, p. 332), Walsh (1901, p. 540; 1921b, pp. 543-544), and Bowley (1901, p. 227). Vogt and Barta (1997, p. 49) correctly observe that this test is a special case of Fisher's (1911, p. 411) proportionality test for quantity indexes which Fisher (1911, p. 405) translated into a test for the price index using the product test (15.3).

16.36 If the price index P satisfies Test T4 and P and Q jointly satisfy the product test (16.17) above, then it is easy to show[23] that Q must satisfy the identity test $Q(p^0,p^1,q,q) = 1$ for all strictly positive vectors p^0,p^1,q. This *constant quantities test* for Q is also somewhat controversial since p^0 and p^1 are allowed to be different.

Homogeneity tests

16.37 The following four tests, T5–T8, restrict the behaviour of the price index P as the scale of any one of the four vectors p^0,p^1,q^0,q^1 changes.

T5: *Proportionality in current prices*:[24]

$$P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1) \text{ for } \lambda > 0.$$

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

16.38 Walsh (1901) and Fisher (1911, p. 418; 1922, p. 420) proposed the related proportionality test $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact Walsh (1901, p. 385) noted that this last test implies the identity test, T3.

16.39 In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T6: *Inverse proportionality in base period prices*:[25]

$$P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1}P(p^0,p^1,q^0,q^1) \quad \text{for } \lambda > 0.$$

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$

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[23] See Vogt (1980, p. 70).

[24] This test was proposed by Walsh (1901, p. 385), Eichhorn and Voeller (1976, p. 24) and Vogt (1980, p. 68).

[25] Eichhorn and Voeller (1976, p. 28) suggested this test.

is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0.

16.40 The following two homogeneity tests can also be regarded as invariance tests.

T7: *Invariance to proportional changes in current quantities*:

$P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 1 quantity vector q^1. Vogt (1980, p. 70) was the first to propose this test[26] and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; i.e., suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then, using the product test (16.17), it can be seen that P must satisfy T7.

T8: *Invariance to proportional changes in base quantities*:[27]

$P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector q^0. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using equation (16.17), the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

16.41 T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors q^0 and q^1.

Invariance and symmetry tests

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[26] Fisher (1911, p. 405) proposed the related test $P(p^0,p^1,q^0,\lambda q^0) = P(p^0,p^1,q^0,q^0) = \sum_{i=1}^{n} p_i^1 q_i^0 \Big/ \sum_{i=1}^{n} p_i^0 q_i^0$.

[27] This test was proposed by Diewert (1992a, p. 216).

16.42 The next five tests, T9–T13, are invariance or symmetry tests. Fisher (1922, pp. 62-63, 458-460) and Walsh (1901, p. 105; 1921b, p. 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, pp. 62-63) spoke of fairness but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index, as is done in paragraphs 16.53 to 16.56. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity reversal test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$$

where p^{t*} denotes a permutation of the components of the vector p^t and q^{t*} denotes the same permutation of the components of q^t for $t = 0,1$. This test is attributable to Fisher (1922, p. 63)[28] and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test, which are considered below.

16.43 The next test asks that the index be invariant to changes in the units of measurement.

T10: *Invariance to changes in the units of measurement* (commensurability test):

$$P(\alpha_1 p_1^0,...,\alpha_n p_n^0;\ \alpha_1 p_1^1,...,\alpha_n p_n^1;\ \alpha_1^{-1} q_1^0,...,\alpha_n^{-1} q_n^0;\ \alpha_1^{-1} q_1^1,...,\alpha_n^{-1} q_n^1) =$$
$$P(p_1^0,...,p_n^0;\ p_1^1,...,p_n^1;\ q_1^0,...,q_n^0;\ q_1^1,...,q_n^1) \text{ for all } \alpha_1 > 0, \ldots, \alpha_n > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test is attributable to Jevons (1863, p. 23) and the Dutch economist Pierson (1896, p. 131), who criticized several index number formulae for not satisfying this fundamental test. Fisher (1911, p. 411) first called this test the *change of units test*; later, Fisher (1922, p. 420) called it the *commensurability test*.

[28] "This [test] is so simple as never to have been formulated. It is merely taken for granted and observed instinctively. Any rule for averaging the commodities must be so general as to apply interchangeably to all of the terms averaged". (Fisher (1922, p. 63))

16.44 The next test asks that the formula be invariant to the period chosen as the base period.

T11: *Time reversal test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio, this test will be satisfied (as are all the other tests listed in this section). When the number of goods is greater than one, many commonly used price indices fail this test; e.g., the Laspeyres (1871) price index, P_L defined by equation (15.5) in Chapter 15, and the Paasche (1874) price index, P_P defined by equation (15.6) in Chapter 15, both fail this fundamental test. The concept of the test is attributable to Pierson (1896, p. 128), who was so upset by the fact that many of the commonly used index number formulae did not satisfy this test that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901, p. 368; 1921b, p. 541) and Fisher (1911, p. 534; 1922, p. 64).

16.45 The next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory. These tests are, however, quite consistent with the weighted stochastic approach to index number theory, discussed later in this chapter.

T12: *Quantity reversal test* (quantity weights symmetry test):

$P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities q^0 and the period 1 quantities q^1 must enter the formula in a symmetric or even-handed manner. Funke and Voeller (1978, p. 3) introduced this test; they called it the *weight property*.

16.46 The next test is the analogue to T12 applied to quantity indices:

T13: *Price reversal test* (price weights symmetry test):[29]

[29] This test was proposed by Diewert (1992a, p. 218).

$$\left(\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0}\right) \Big/ P(p^0, p^1, q^0, q^1) = \left(\frac{\sum_{i=1}^{n} p_i^0 q_i^1}{\sum_{i=1}^{n} p_i^1 q_i^0}\right) \Big/ P(p^1, p^0, q^0, q^1) \qquad (16.18)$$

Thus if we use equation (16.17) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q:

$$Q(p^0, p^1, q^0, q^1) = Q(p^1, p^0, q^0, q^1) \qquad (16.19)$$

That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

Mean value tests

16.47 The next three tests, T14–T16, are mean value tests.

T14: *Mean value test for prices*:[30]

$$\min_i (p_i^1 / p_i^0 \ : i \ = \ 1,...,n) \leq P(p^0, p^1, q^0, q^1) \leq \max_i (p_i^1 / p_i^0 \ : i \ = \ 1,...,n) \qquad (16.20)$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be interpreted as some sort of an average of the n price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

16.48 The next test is the analogue to T14 applied to quantity indices:

T15: *Mean value test for quantities*:[31]

$$\min_i (q_i^1 / q_i^0 \ : i = \ 1,...,n) \leq \frac{(V^1 / V^0)}{P(p^0, p^1, q^0, q^1)} \leq \max_i (q_i^1 / q_i^0 \ : i \ = \ 1,...,n) \qquad (16.21)$$

where V^t is the period t value for the aggregate defined by equation (16.1). Using the product test (16.17) to define the quantity index Q in terms of the price index P, it can be seen that T15 is equivalent to the following property for the associated quantity index Q:

$$\min_i (q_i^1/q_i^0 \ : i = 1,...,n) \ \leq Q(p^0, p^1, q^0, q^1) \leq \max_i \ (q_i^1/q_i^0 \ : i = 1,...,n) \qquad (16.22)$$

[30] This test seems to have been first proposed by Eichhorn and Voeller (1976, p. 10).

[31] This test was proposed by Diewert (1992a, p. 219).

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_i^1/q_i^0 of the individual quantities.

16.49 In paragraphs 15.18 to 15.32 of Chapter 15, it was argued that it is very reasonable to take an average of the Laspeyres and Paasche price indices as a single "best" measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres bounding test*:[32]

The price index P lies between the Laspeyres and Paasche indices, P_L and P_P, defined by equations (15.5) and (15.6) in Chapter 15.

A test could be proposed where the implicit quantity index Q that corresponds to P via equation (16.17) is to lie between the Laspeyres and Paasche quantity indices, Q_P and Q_L, defined by equations (15.10) and (15.11) in Chapter 15. However, the resulting test turns out to be equivalent to test T16.

Monotonicity tests

16.50 The final four tests, T17–T20, are monotonicity tests; i.e., how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors p^0 and p^1 increases or as any component of the two quantity vectors q^0 and q^1 increases?

T17: *Monotonicity in current prices*:

$P(p^0,p^1,q^0,q^1) < P(p^0,p^2,q^0,q^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1. This property was proposed by Eichhorn and Voeller (1976, p. 23) and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in base prices*: $P(p^0,p^1,q^0,q^1) > P(p^2,p^1,q^0,q^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0. This very reasonable property was also proposed by Eichhorn and Voeller (1976, p. 23).

T19: *Monotonicity in current quantities*: if $q^1 < q^2$, then

[32] Bowley (1901, p. 227) and Fisher (1922, p. 403) both endorsed this property for a price index.

$$\left(\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0}\right) \Big/ P(p^0, p^1, q^0, q^1) < \left(\frac{\sum_{i=1}^{n} p_i^1 q_i^2}{\sum_{i=1}^{n} p_i^0 q_i^0}\right) \Big/ P(p^0, p^1, q^0, q^2) \qquad (16.23)$$

T20: *Monotonicity in base quantities*: if $q^0 < q^2$, then

$$\left(\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0}\right) \Big/ P(p^0, p^1, q^0, q^1) > \left(\frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^2}\right) \Big/ P(p^0, p^1, q^2, q^1). \qquad (16.24)$$

16.51 Let Q be the implicit quantity index that corresponds to P using equation (16.17). Then it is found that T19 translates into the following inequality involving Q:

$$Q(p^0, p^1, q^0, q^1) < Q(p^0, p^1, q^0, q^2) \quad \text{if} \quad q^1 < q^2 \qquad (16.25)$$

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

$$Q(p^0, p^1, q^0, q^1) > Q(p^0, p^1, q^2, q^1) \quad \text{if} \quad q^0 < q^2 \qquad (16.26)$$

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are attributable to Vogt (1980, p. 70).

16.52 This concludes the listing of tests. The next section offers an answer to the question of whether any index number formula $P(p^0,p^1,q^0,q^1)$ exists that can satisfy all 20 tests.

The Fisher ideal index and the test approach

16.53 It can be shown that the only index number formula $P(p^0,p^1,q^0,q^1)$ which satisfies tests T1-T20 is the Fisher ideal price index P_F defined as the geometric mean of the Laspeyres and Paasche indices:[33]

$$P_F(p^0, p^1, q^0, q^1) \equiv \left\{P_L(p^0, p^1, q^0, q^1) \;\; P_P(p^0, p^1, q^0, q^1)\right\}^{1/2} \qquad (16.27)$$

16.54 It is relatively straightforward to show that the Fisher index satisfies all 20 tests. The more difficult part of the proof is to show that the Fisher index is the *only* index number formula that satisfies these tests. This part of the proof follows from the fact that, if P

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33 See Diewert (1992a, p. 221).

satisfies the positivity test T1 and the three reversal tests, T11-T13, then P must equal P_F. To see this, rearrange the terms in the statement of test T13 into the following equation:

$$\frac{\sum_{i=1}^{n} p_i^1 q_i^1 \Big/ \sum_{i=1}^{n} p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^1 \Big/ \sum_{i=1}^{n} p_i^1 q_i^0} = \frac{P(p^0,p^1,q^0,q^1)}{P(p^1,p^0,q^0,q^1)}$$

$$= \frac{P(p^0,p^1,q^0,q^1)}{P(p^1,p^0,q^1,q^0)} \quad \text{using T12, the quantity reversal test} \qquad (16.28)$$

$$= P(p^0,p^1,q^0,q^1)P(p^0,p^1,q^0,q^1) \quad \text{using T11, the time reversal test.}$$

Now take positive square roots of both sides of equation (16.28). It can be seen that the left-hand side of the equation is the Fisher index $P_F(p^0,p^1,q^0,q^1)$ defined by equation (16.27) and the right-hand side is $P(p^0,p^1,q^0,q^1)$. Thus if P satisfies T1, T11, T12 and T13, it must equal the Fisher ideal index P_F.

16.55 The quantity index that corresponds to the Fisher price index using the product test (16.17) is Q_F , the Fisher quantity index, defined by equation (15.14) in Chapter 15.

16.56 It turns out that P_F satisfies yet another test, T21, which was Fisher's (1921, p. 534; 1922, pp. 72-81) third reversal test (the other two being T9 and T11):

T21: *Factor reversal test* (functional form symmetry test):

$$P(p^0,p^1,q^0,q^1)P(q^0,q^1,p^0,p^1) = \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0} \qquad (16.29)$$

A justification for this test is the following: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then, if the roles of prices and quantities are reversed, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, V^1 / V^0 . The second part of this argument does not seem to be valid, and thus many researchers over the years have objected to the factor reversal test. Nevertheless, if T21 is accepted as a basic test, Funke and Voeller (1978, p. 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ which satisfies T1 (positivity), T11 (time reversal test), T12 (quantity reversal test) and T21 (factor reversal test) is the Fisher ideal index P_F defined by equation (16.27). Thus the price reversal test T13 can be replaced by the factor reversal test in order to obtain a minimal set of four tests that lead to the Fisher price index.[34]

The test performance of other indices

16.57 The Fisher price index P_F satisfies all 20 of the tests T1–T20 listed above. Which tests do other commonly used price indices satisfy? Recall the Laspeyres index P_L defined by equation (15.5), the Paasche index P_P defined by equation (15.6), the Walsh index P_W defined by equation (15.19) and the Törnqvist index P_T defined by equation (15.81) in Chapter 15.

16.58 Straightforward computations show that the Paasche and Laspeyres price indices, P_L and P_P, fail only the three reversal tests, T11, T12 and T13. Since the quantity and price reversal tests, T12 and T13, are somewhat controversial and hence can be discounted, the test performance of P_L and P_P seems at first sight to be quite good. The failure of the time reversal test, T11, is nevertheless a severe limitation associated with the use of these indices.

16.59 The Walsh price index, P_W, fails four tests: T13, the price reversal test; T16, the Paasche and Laspeyres bounding test; T19, the monotonicity in current quantities test; and T20, the monotonicity in base quantities test.

16.60 Finally, the Törnqvist price index P_T fails nine tests: T4 (the fixed basket test), the quantity and price reversal tests T12 and T13, T15 (the mean value test for quantities), T16 (the Paasche and Laspeyres bounding test) and the four monotonicity tests T17 to T20. Thus the Törnqvist index is subject to a rather high failure rate from the viewpoint of this axiomatic approach to index number theory.[35]

16.61 The tentative conclusion that can be drawn from the above results is that, from the viewpoint of this particular bilateral test approach to index numbers, the Fisher ideal price index P_F appears to be "best" since it satisfies all 20 tests. The Paasche and Laspeyres indices

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[34] Other characterizations of the Fisher price index can be found in Funke and Voeller (1978) and Balk (1985; 1995).

[35] It is shown in Chapter 19, however, that the Törnqvist index approximates the Fisher index quite closely using "normal" time series data that are subject to relatively smooth trends. Hence, under these circumstances, the Törnqvist index can be regarded as passing the 20 tests to a reasonably high degree of approximation.

are next best if we treat each test as being equally important. Both of these indices, however, fail the very important time reversal test. The remaining two indices, the Walsh and Törnqvist price indices, both satisfy the time reversal test but the Walsh index emerges as being "better" since it passes 16 of the 20 tests whereas the Törnqvist only satisfies 11 tests.[36]

The additivity test

16.62 There is an additional test that many national income accountants regard as very important: the *additivity test*. This is a test or property that is placed on the implicit quantity index $Q(p^0,p^1,q^0,q^1)$ that corresponds to the price index $P(p^0,p^1,q^0,q^1)$ using the product test (16.17). This test states that the implicit quantity index has the following form:

$$Q(p^0,p^1,q^0,q^1) = \frac{\sum_{i=1}^{n} p_i^* q_i^1}{\sum_{m=1}^{n} p_m^* q_m^0} \qquad (16.30)$$

where the common across-periods *price* for commodity i, p_i* for $i = 1,\ldots,n$, can be a function of all 4n prices and quantities pertaining to the two periods or situations under consideration, p^0,p^1,q^0,q^1. In the literature on making multilateral comparisons (i.e., comparisons between more than two situations), it is quite common to assume that the quantity comparison between any two regions can be made using the two regional quantity vectors, q^0 and q^1, and a common reference price vector, $p^* \equiv (p_1^*,\ldots,p_n^*)$.[37]

16.63 Obviously, different versions of the additivity test can be obtained if further restrictions are placed on precisely which variables each reference price p_i* depends. The simplest such restriction is to assume that each p_i* depends only on the commodity i prices pertaining to each of the two situations under consideration, p_i^0 and p_i^1. If it is further assumed that the functional form for the weighting function is the same for each commodity,

[36] This assertion needs to be qualified: there are many other tests that we have not discussed, and price statisticians might hold different opinions regarding the importance of satisfying various sets of tests. Other tests are discussed by Auer (2001; 2002), Eichhorn and Voeller (1976), Balk (1995) and Vogt and Barta (1997), among others. It is shown in paragraphs 16.101 to 16.135 that the Törnqvist index is ideal when considered under a different set of axioms.

[37] Hill (1993, p. 395-397) termed such multilateral methods *the block approach* while Diewert (1996a, pp. 250-251) used the term *average price approaches*. Diewert (1999b, p. 19) used the term *additive multilateral system*. For axiomatic approaches to multilateral index number theory, see Balk (1996a; 2001) and Diewert (1999b).

so that $p_i^* = m(p_i^0,p_i^1)$ for $i = 1,\ldots,n$, then we are led to the *unequivocal quantity index* postulated by Knibbs (1924, p. 44).

16.64 The theory of the *unequivocal quantity index* (or the *pure quantity index*)[38] parallels the theory of the pure price index outlined in paragraphs 15.24 to 15.32 of Chapter 15. An outline of this theory is given here. Let the pure quantity index Q_K have the following functional form:

$$Q_K(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} q_i^1 m(p_i^0,p_i^1)}{\sum_{k=1}^{n} q_k^0 m(p_k^0,p_k^1)} \qquad (16.31)$$

It is assumed that the price vectors p^0 and p^1 are strictly positive and the quantity vectors q^0 and q^1 are non-negative but have at least one positive component.[39] The problem is to determine the functional form for the averaging function m if possible. To do this, it is necessary to impose some tests or properties on the pure quantity index Q_K. As was the case with the pure price index, it is very reasonable to ask that the quantity index satisfy the *time reversal test*:

$$Q_K(p^1,p^0,q^1,q^0) = \frac{1}{Q_K(p^0,p^1,q^0,q^1)} \qquad (16.32)$$

16.65 As was the case with the theory of the unequivocal price index, it can be seen that if the unequivocal quantity index Q_K is to satisfy the time reversal test (16.32), the mean function in equation (16.31) must be *symmetric*. It is also asked that Q_K satisfy the following *invariance to proportional changes in current prices test*.

$$Q_K(p^0,\lambda p^1,q^0,q^1) = Q_K(p^0,p^1,q^0,q^1) \text{ for all } p^0,p^1,q^0,q^1 \text{ and all } \lambda > 0. \qquad (16.33)$$

16.66 The idea behind this invariance test is this: the quantity index $Q_K(p^0,p^1,q^0,q^1)$ should depend only on the *relative* prices in each period and it should not depend on the amount of inflation between the two periods. Another way to interpret test (16.33) is to look at what the test implies for the corresponding implicit price index, P_{IK}, defined using the product test

[38] Diewert (2001) used this term.

[39] It is assumed that $m(a,b)$ has the following two properties: $m(a,b)$ is a positive and continuous function, defined for all positive numbers a and b, and $m(a,a) = a$ for all $a > 0$.

(16.17). It can be shown that if Q_K satisfies equation (16.33), then the corresponding implicit price index P_{IK} will satisfy test T5 above, the *proportionality in current prices test*. The two tests, (16.32) and (16.33), determine the precise functional form for the pure quantity index Q_K defined by equation (16.31): the *pure quantity index* or Knibbs' *unequivocal quantity index* Q_K must be the Walsh quantity index Q_W[40] defined by:

$$Q_W(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} q_i^1 \sqrt{p_i^0 p_i^1}}{\sum_{k=1}^{n} q_k^0 \sqrt{p_k^0 p_k^1}} \quad (16.34)$$

16.67 Thus with the addition of two tests, the pure price index P_K must be the Walsh price index P_W defined by equation (15.19) in Chapter 15 and with the addition of the same two tests (but applied to quantity indices instead of price indices), the pure quantity index Q_K must be the Walsh quantity index Q_W defined by equation (16.34). Note, however, that the product of the Walsh price and quantity indices is *not* equal to the expenditure ratio, V^1/V^0. Thus believers in the pure or unequivocal price and quantity index concepts have to choose one of these two concepts; they both cannot apply simultaneously.[41]

16.68 If the quantity index $Q(p^0,p^1,q^0,q^1)$ satisfies the additivity test (16.30) for some price weights p_i^*, then the percentage change in the quantity aggregate, $Q(p^0,p^1,q^0,q^1) - 1$, can be rewritten as follows:

$$Q(p^0,p^1,q^0,q^1) - 1 = \frac{\sum_{i=1}^{n} p_i^* q_i^1}{\sum_{m=1}^{n} p_m^* q_m^0} - 1 = \frac{\sum_{i=1}^{n} p_i^* q_i^1 - \sum_{m=1}^{n} p_m^* q_m^0}{\sum_{m=1}^{n} p_m^* q_m^0} = \sum_{i=1}^{n} w_i (q_i^1 - q_i^0) \quad (16.35)$$

where the *weight* for commodity i, w_i, is defined as

$$w_i \equiv \frac{p_i^*}{\sum_{m=1}^{n} p_m^* q_m^0}; \quad i = 1,\ldots,n \quad (16.36)$$

[40] This is the quantity index that corresponds to the price index 8 defined by Walsh (1921a, p. 101), see equation (15.19)

[41] Knibbs (1924) did not notice this point.

Note that the change in commodity i going from situation 0 to situation 1 is $q_i^1 - q_i^0$. Thus the ith term on the right-hand side of equation (16.35) is the contribution of the change in commodity i to the overall percentage change in the aggregate going from period 0 to 1. Business analysts often want statistical agencies to provide decompositions such as equation (16.35) so that they can decompose the overall change in an aggregate into sector-specific components of change.[42] Thus there is a demand on the part of users for additive quantity indices.

16.69 For the Walsh quantity index defined by equation (16.34), the ith weight is

$$w_{W_i} \equiv \frac{\sqrt{p_i^0 p_i^1}}{\sum_{m=1}^{n} q_m^0 \sqrt{p_m^0 p_m^1}}; \quad i = 1,\ldots,n \tag{16.37}$$

Thus the Walsh quantity index Q_W has a percentage decomposition into component changes of the form of equation (16.35), where the weights are defined by equation (16.37).

16.70 It turns out that the Fisher quantity index Q_F, defined by equation (15.14) in Chapter 15, also has an additive percentage change decomposition of the form given by equation (16.35).[43] The ith weight w_{F_i} for this Fisher decomposition is rather complicated and depends on the Fisher quantity index $Q_F(p^0,p^1,q^0,q^1)$ as follows:[44]

$$w_{F_i} \equiv \frac{w_i^0 + (Q_F)^2 w_i^1}{1 + Q_F}; \ i = 1,\ldots,n \tag{16.38}$$

where Q_F is the value of the Fisher quantity index, $Q_F(p^0,p^1,q^0,q^1)$, and the period t normalized price for commodity i, w_i^t, is defined as the period t price p_i^t divided by the period t expenditure on the aggregate:

[42] Business and government analysts also often demand an analogous decomposition of the change in price aggregate into sector-specific components that add up.

[43] The Fisher quantity index also has an additive decomposition of the type defined by equation (16.30) attributable to Van Ijzeren (1987, p. 6). The ith reference price p_i^* is defined as $p_i^* \equiv \left[(1/2)p_i^0 + (1/2)p_i^1\right] / P_F\left(p^0 p^1 q^0 q^1\right)$ for $i = 1,\ldots,n$ and where P_F is the Fisher price index. This decomposition was also independently derived by Dikhanov (1997). The Van Ijzeren decomposition for the Fisher quantity index is currently being used by the US Bureau of Economic Analysis; see Moulton and Seskin (1999, p. 16) and Ehemann, Katz and Moulton (2002).

[44] This decomposition was obtained by Diewert (2002a) and Reinsdorf, Diewert and Ehemann (2002). For an economic interpretation of this decomposition, see Diewert (2002a).

$$w_i^t \equiv \frac{p_i^t}{\sum_{m=1}^{n} p_m^t q_m^t}; \quad t = 0,1 \,; \quad i = 1,\ldots,n \tag{16.39}$$

16.71 Using the weights w_{F_i} defined by equations (16.38) and (16.39), the following exact decomposition is obtained for the Fisher ideal quantity index:

$$Q_F(p^0, p^1, q^0, q^1) - 1 = \sum_{i=1}^{n} w_{F_i}(q_i^1 - q_i^0) \tag{16.40}$$

Thus the Fisher quantity index has an additive percentage change decomposition.[45]

16.72 Because of the symmetric nature of the Fisher price and quantity indices, it can be seen that the Fisher price index P_F defined by equation (16.27) also has the following additive percentage change decomposition:

$$P_F(p^0, p^1, q^0, q^1) - 1 = \sum_{i=1}^{n} v_{F_i}(p_i^1 - p_i^0) \tag{16.41}$$

where the commodity i weight v_{F_i} is defined as

$$v_{F_i} \equiv \frac{v_i^0 + (P_F)^2 v_i^1}{1 + P_F}; \; i = 1,\ldots,n \tag{16.42}$$

where P_F is the value of the Fisher price index, $P_F(p^0,p^1,q^0,q^1)$, and the period t normalized quantity for commodity i, v_i^t, is defined as the period t quantity q_i^t divided by the period t expenditure on the aggregate:

$$v_i^t \equiv \frac{q_i^t}{\sum_{m=1}^{n} p_m^t q_m^t}; \quad t = 0,1 \,; \quad i = 1,\ldots,n \tag{16.43}$$

16.73 The above results show that the Fisher price and quantity indices have exact additive decompositions into components that give the contribution to the overall change in the price (or quantity) index of the change in each price (or quantity).

[45] To verify the exactness of the decomposition, substitute equation (16.38) into equation (16.40) and solve the resulting equation for Q_F. It is found that the solution is equal to Q_F defined by equation (15.14) in Chapter 15.

The stochastic approach to price indices

The early unweighted stochastic approach

16.74 The stochastic approach to the determination of the price index can be traced back to the work of Jevons (1863; 1865) and Edgeworth (1888) over 100 years ago.[46] The basic idea behind the (unweighted) stochastic approach is that each price relative, p_i^1/p_i^0 for $i = 1,2,\ldots,n$ can be regarded as an estimate of a common inflation rate α between periods 0 and 1.[47] It is assumed that

$$\frac{p_i^1}{p_i^0} = \alpha + \varepsilon_i\,; \;\; i = 1,2,\ldots,n \qquad (16.44)$$

where α is the common inflation rate and the ε_i are random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for α is the Carli (1764) price index P_C defined as

$$P_C(p^0, p^1) \equiv \sum_{i=1}^{n} \frac{1}{n} \frac{p_i^1}{p_i^0} \qquad (16.45)$$

A drawback of the Carli price index is that it does not satisfy the time reversal test, i.e., $P_C(p^1,p^0) \neq 1/\, P_C(p^0,p^1)$.[48]

16.75 Now change the stochastic specification and assume that the logarithm of each price relative, $\ln(p_i^1/p_i^0)$, is an unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. The counterpart to equation (16.44) is:

$$\ln\left(\frac{p_i^1}{p_i^0}\right) = \beta + \varepsilon_i\,; \quad i = 1,2,\ldots,n \qquad (16.46)$$

where $\beta \equiv \ln\alpha$ and the ε_i are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the

46 For references to the literature, see Diewert (1993a, pp. 37-38; 1995a; 1995b).

47 "In drawing our averages the independent fluctuations will more or less destroy each other; the one required variation of gold will remain undiminished" (Jevons (1863, p. 26)).

48 In fact, Fisher (1922, p. 66) noted that $P_C(p^0,p^1)P_C(p^1,p^0) > 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; i.e., Fisher showed that the Carli index has a definite upward bias. He urged statistical agencies not to use this formula. Walsh (1901, pp. 331, 530) also discovered this result for the case $n = 2$.

geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α[49] is the Jevons (1865) price index P_J defined as follows:

$$P_J(p^0, p^1) \equiv \prod_{i=1}^{n} \sqrt[n]{\frac{p_i^1}{p_i^0}} \qquad (16.47)$$

16.76 The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C. Both the Jevons and Carli price indices nevertheless suffer from a fatal flaw: each price relative p_i^1/p_i^0 is regarded as being equally important and is given an equal weight in the index number formulae (16.45) and (16.47). John Maynard Keynes was particularly critical of this unweighted stochastic approach to index number theory.[50] He directed the following criticism towards this approach, which was vigorously advocated by Edgeworth (1923):

> Nevertheless I venture to maintain that such ideas, which I have endeavoured to expound above as fairly and as plausibly as I can, are root-and-branch erroneous. The "errors of observation", the "faulty shots aimed at a single bull's eye" conception of the index number of prices, Edgeworth's "objective mean variation of general prices", is the result of confusion of thought. There is no bull's eye. There is no moving but unique centre, to be called the general price level or the objective mean variation of general prices, round which are scattered the moving price levels of individual things. There are all the various, quite definite, conceptions of price levels of composite commodities appropriate for various purposes and inquiries which have been scheduled above, and many others too. There is nothing else. Jevons was pursuing a mirage.
>
> What is the flaw in the argument? In the first place it assumed that the fluctuations of individual prices round the "mean" are "random" in the sense required by the theory of the combination of independent observations. In this theory the divergence of one "observation" from the true position is assumed to have no influence on the divergences of other "observations". But in the case of prices, a movement in

[49] Greenlees (1999) pointed out that although $(1/n)\sum_{i=1}^{n} \ln(p_i^1/p_i^0)$ is an unbiased estimator for β, the corresponding exponential of this estimator, P_J defined by equation (16.47), will generally not be an unbiased estimator for α under our stochastic assumptions. To see this, let $x_i = \ln p_i^1/p_i^0$. Taking expectations, we have: $Ex_i = \beta = \ln \alpha$. Define the positive, convex function f of one variable x by $f(x) \equiv e^x$. By Jensen's (1906) inequality, $Ef(x) \geq f(Ex)$. Letting x equal the random variable x_i, this inequality becomes: $E(p_i^1/p_i^0) = Ef(x_i) \geq f(Ex_i) = f(\beta) = e^\beta = e^{\ln \alpha} = \alpha$. Thus for each n, $E(p_i^1/p_i^0) \geq \alpha$, and it can be seen that the Jevons price index will generally have an upward bias under the usual stochastic assumptions.

[50] Walsh (1901, p. 83) also stressed the importance of proper weighting according to the economic importance of the commodities in the periods being compared: "But to assign uneven weighting with approximation to the relative sizes, either over a long series of years or for every period separately, would not require much additional trouble; and even a rough procedure of this sort would yield results far superior to those yielded by even weighting. It is especially absurd to refrain from using roughly reckoned uneven weighting on the ground that it is not accurate, and instead to use even weighting, which is much more inaccurate."

> the price of one commodity necessarily influences the movement in the prices of other commodities, whilst the magnitudes of these compensatory movements depend on the magnitude of the change in expenditure on the first commodity as compared with the importance of the expenditure on the commodities secondarily affected. Thus, instead of "independence", there is between the "errors" in the successive "observations" what some writers on probability have called "connexity", or, as Lexis expressed it, there is "sub-normal dispersion".
>
> We cannot, therefore, proceed further until we have enunciated the appropriate law of connexity. But the law of connexity cannot be enunciated without reference to the relative importance of the commodities affected—which brings us back to the problem that we have been trying to avoid, of weighting the items of a composite commodity (Keynes (1930, pp. 76-77)).

The main point Keynes seemed to be making in the above quotation is that prices in the economy are not independently distributed from each other and from quantities. In current macroeconomic terminology, Keynes can be interpreted as saying that a macroeconomic shock will be distributed across all prices and quantities in the economy through the normal interaction between supply and demand; i.e., through the workings of the general equilibrium system. Thus Keynes seemed to be leaning towards the economic approach to index number theory (even before it was developed to any great extent), where quantity movements are functionally related to price movements. A second point that Keynes made in the above quotation is that there is no such thing as *the* inflation rate; there are only price changes that pertain to well-specified sets of commodities or transactions; i.e., the domain of definition of the price index must be carefully specified.[51] A final point that Keynes made is that price movements must be weighted by their economic importance; i.e., by quantities or expenditures.

16.77 In addition to the above theoretical criticisms, Keynes also made the following strong empirical attack on Edgeworth's unweighted stochastic approach:

> The Jevons–Edgeworth "objective mean variation of general prices", or "indefinite" standard, has generally been identified, by those who were not as alive as Edgeworth himself was to the subtleties of the case, with the purchasing power of money—if only for the excellent reason that it was difficult to visualise it as anything else. And since any respectable index number, however weighted, which covered a fairly large number of commodities could, in accordance with the argument, be regarded as a

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[51] See paragraphs 15.7 to 15.17 in Chapter 15 for additional discussion on this point.

fair approximation to the indefinite standard, it seemed natural to regard any such index as a fair approximation to the purchasing power of money also.

Finally, the conclusion that all the standards "come to much the same thing in the end" has been reinforced "inductively" by the fact that rival index numbers (all of them, however, of the wholesale type) have shown a considerable measure of agreement with one another in spite of their different compositions … On the contrary, the tables given above (pp. 53, 55) supply strong presumptive evidence that over long period as well as over short period the movements of the wholesale and of the consumption standards respectively are capable of being widely divergent (Keynes (1930, pp. 80-81)).

In the above quotation, Keynes noted that the proponents of the unweighted stochastic approach to price change measurement were comforted by the fact that all of the then existing (unweighted) indices of wholesale prices showed broadly similar movements. Keynes showed empirically, however, that his wholesale price indices moved quite differently from his consumer price indices.

16.78 In order to overcome the above criticisms of the unweighted stochastic approach to index numbers, it is necessary to:

- have a definite domain of definition for the index number;
- weight the price relatives by their economic importance.[52]

Alternative methods of weighting are discussed in the following sections.

The weighted stochastic approach

16.79 Walsh (1901, pp. 88-89) seems to have been the first index number theorist to point out that a sensible stochastic approach to measuring price change means that individual price relatives should be weighted according to their economic importance or their *transactions value* in the two periods under consideration:

> It might seem at first sight as if simply every price quotation were a single item, and since every commodity (any kind of commodity) has one price-quotation attached to it, it would seem as if price-variations of every kind of commodity were the single item in question. This is the way the question struck the first inquirers into price-variations, wherefore they used simple averaging with even weighting. But a price-quotation is the quotation of the price of a generic name for many articles; and

[52] Walsh (1901, pp. 82-90; 1921a, pp. 82-83) also objected to the lack of weighting in the unweighted stochastic approach to index number theory.

one such generic name covers a few articles, and another covers many. … A single price-quotation, therefore, may be the quotation of the price of a hundred, a thousand, or a million dollar's worths, of the articles that make up the commodity named. Its weight in the averaging, therefore, ought to be according to these money-unit's worth (Walsh (1921a, pp. 82-83)).

But Walsh did not give a convincing argument on exactly how these economic weights should be determined.

16.80 Henri Theil (1967, pp. 136-137) proposed a solution to the lack of weighting in the Jevons index, P_J defined by equation (16.47). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the *i*th price relative is equal to $s_i^0 \equiv p_i^0 q_i^0 \big/ \sum_{k=1}^n p_k^0 q_k^0$, the period 0 expenditure share for commodity *i*. Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{i=1}^n s_i^0 \ln(p_i^1 / p_i^0)$.[53] Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{i=1}^n s_i^1 \ln(p_i^1 / p_i^0)$.[54]

16.81 Each of these measures of overall logarithmic price change seems equally valid, so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change. Theil[55] argued that a "nice" symmetric index number formula can be obtained if the probability of selection for the *n*th price relative is made equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity *n.*

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[53] In Chapter 19, this index is called the *geometric Laspeyres index*, P_{GL}. Vartia (1978, p. 272) referred to this index as the *logarithmic Laspeyres index*. Yet another name for the index is the *base weighted geometric index*.

[54] In Chapter 19, this index is called the *geometric Paasche index*, P_{GP}. Vartia (1978, p. 272) referred to this index as the *logarithmic Paasche index*. Yet another name for the index is the *current period weighted geometric index*.

[55] "The price index number defined in (1.8) and (1.9) uses the *n* individual logarithmic price differences as the basic ingredients. They are combined linearly by means of a two-stage random selection procedure: First, we give each region the same chance ½ of being selected, and second, we give each dollar spent in the selected region the same chance ($1/m_a$ or $1/m_b$) of being drawn" (Theil (1967, p. 138)). The indexes (1.8) and (1.9) are the geometric Laspeyres and Paasche indexes.

Using these probabilities of selection, Theil's final measure of overall logarithmic price change was

$$\ln P_T(p^0, p^1, q^0, q^1) \equiv \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1) \ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (16.48)$$

Note that the index P_T defined by equation (16.48) is equal to the Törnqvist index defined by equation (15.81) in Chapter 15.

16.82 A statistical interpretation of the right-hand side of equation (16.48) can be given. Define the *i*th logarithmic price ratio r_i by:

$$r_i \equiv \ln\left(\frac{p_i^1}{p_i^0}\right) \quad \text{for } i = 1,\ldots,n \qquad (16.49)$$

Now define the discrete random variable, *R* say, as the random variable which can take on the values r_i with probabilities $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for $i = 1,\ldots,n$. Note that, since each set of expenditure shares, s_i^0 and s_i^1, sums to one over *i*, the probabilities ρ_i will also sum to one. It can be seen that the expected value of the discrete random variable *R* is

$$\begin{aligned} E[R] &\equiv \sum_{i=1}^{n} \rho_i r_i = \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1) \ln\left(\frac{p_i^1}{p_i^0}\right) \\ &= \ln P_T(p^0, p^1, q^0, q^1). \end{aligned} \qquad (16.50)$$

Thus the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the *n* discrete price ratios in this domain of definition are weighted according to Theil's probability weights, $\rho_i \equiv (1/2)[s_i^0 + s_i^1]$ for $i = 1,\ldots,n$.

16.83 Taking antilogs of both sides of equation (16.48), the Törnqvist (1936; 1937) Theil price index, P_T, is obtained.[56] This index number formula has a number of good properties. In particular, P_T satisfies the proportionality in current prices test T5 and the time reversal test T11, discussed above. These two tests can be used to justify Theil's (arithmetic) method of forming an average of the two sets of expenditure shares in order to obtain his probability

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[56] The sampling bias problem studied by Greenlees (1999) (see footnote 49 above) does not occur in the present context because there is no sampling involved in definition (16.50): the sum of the $p_i^t q_i^t$ over *i* for each period *t* is assumed to equal the value aggregate V^t for period *t*.

weights, $\rho_i \equiv (1/2)[\, s_i^0 + s_i^1]$ for $i = 1,\ldots,n$. Consider the following *symmetric mean class of logarithmic index number formulae*:

$$\ln P_S(p^0, p^1, q^0, q^1) \equiv \sum_{i=1}^{n} m(s_i^0, s_i^1) \ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (16.51)$$

where $m(s_i^0, s_i^1)$ is a positive function of the period 0 and 1 expenditure shares on commodity i, s_i^0 and s_i^1 respectively. In order for P_S to satisfy the time reversal test, it is necessary that the function m be symmetric. Then it can be shown[57] that for P_S to satisfy test T5, m must be the arithmetic mean. This provides a reasonably strong justification for Theil's choice of the mean function.

16.84 The stochastic approach of Theil has another "nice" symmetry property. Instead of considering the distribution of the logarithmic price ratios $r_i = \ln p_i^1/p_i^0$, we could also consider the distribution of the logarithms of the *reciprocals* of the price ratios, say:

$$t_i \equiv \ln\frac{p_i^0}{p_i^1} = \ln\left(\frac{p_i^1}{p_i^0}\right)^{-1} = -\ln\frac{p_i^1}{p_i^0} = -r_i \quad \text{for } i = 1,\ldots,n \qquad (16.52)$$

The symmetric probability, $\rho_i \equiv (1/2)[\, s_i^0 + s_i^1]$, can still be associated with the i^{th} logarithmic reciprocal price ratio t_i for $i = 1,\ldots,n$. Now define the discrete random variable, T say, as the random variable which can take on the values t_i with probabilities $\rho_i \equiv (1/2)[\, s_i^0 + s_i^1]$ for $i = 1,\ldots,n$. It can be seen that the expected value of the discrete random variable T is

$$\begin{aligned} E[T] &\equiv -\sum_{i=1}^{n} \rho_i r_i \\ &= E[R] \quad \text{using (16.50)} \qquad (16.53) \\ &= -\sum_{i=1}^{n} \rho_i t_i \quad \text{using (16.52)} \\ &= -\ln P_T(p^0, p^1, q^0, q^1). \end{aligned}$$

Thus it can be seen that the distribution of the random variable T is the same as the distribution of the random variable R, but T takes on values of opposite sign from R. Hence it does not matter whether the distribution of the original logarithmic price ratios, $r_i \equiv \ln p_i^1/p_i^0$, is considered or the distribution of the log of their reciprocals, $t_i \equiv \ln p_i^0/p_i^1$, is considered: essentially the same stochastic theory is obtained.

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[57] See Diewert (2000) and Balk and Diewert (2001).

16.85 It is possible to consider weighted stochastic approaches to index number theory where the distribution of the price ratios, p_i^1/p_i^0, is considered rather than the distribution of the logarithmic price ratios, $\ln p_i^1/p_i^0$. Thus, again following in the footsteps of Theil, suppose that price relatives are drawn at random in such a way that each dollar of expenditure in the *base period* has an equal chance of being selected. Then the probability that the *i*th price relative will be drawn is equal to s_i^0, the period 0 expenditure share for commodity *i*. Thus the overall mean (period 0 weighted) price change is:

$$P_L(p^0, p^1, q^0, q^1) = \sum_{i=1}^{n} s_i^0 \frac{p_i^1}{p_i^0} \qquad (16.54)$$

which turns out to be the Laspeyres price index, P_L. This stochastic approach is the natural one for studying *sampling problems* associated with implementing a Laspeyres price index.

16.86 Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) price change equal to:

$$P_{PAL}(p^0, p^1, q^0, q^1) = \sum_{i=1}^{n} s_i^1 \frac{p_i^1}{p_i^0} \qquad (16.55)$$

This is known as the Palgrave (1886) index number formula.[58]

16.87 It can be verified that neither the Laspeyres nor Palgrave price indices satisfy the time reversal test, T11. Thus, again following in the footsteps of Theil, it might be attempted to obtain a formula that satisfied the time reversal test by taking a symmetric average of the two sets of shares. Thus consider the following class of *symmetric mean index number formulae*:

$$P_m(p^0, p^1, q^0, q^1) \equiv \sum_{i=1}^{n} m(s_i^0, s_i^1) \frac{p_i^1}{p_i^0} \qquad (16.56)$$

where $m(s_i^0, s_i^1)$ is a symmetric function of the period 0 and 1 expenditure shares for commodity *i*, s_i^0 and s_i^1 respectively. In order to interpret the right hand-side of equation (16.56) as an expected value of the price ratios p_i^1/p_i^0, it is necessary that

$$\sum_{i=1}^{n} m(s_i^0, s_i^1) = 1 \qquad (16.57)$$

[58] It is formula number 9 in Fisher's (1922, p. 466) listing of index number formulae.

In order to satisfy equation (16.57), however, m must be the arithmetic mean.[59] With this choice of m, equation (16.56) becomes the following (unnamed) index number formula, P_u:

$$P_u(p^0, p^1, q^0, q^1) \equiv \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1)\frac{p_i^1}{p_i^0} \qquad (16.58)$$

Unfortunately, the unnamed index P_u does not satisfy the time reversal test either.[60]

16.88 Instead of considering the distribution of the price ratios, p_i^1/p_i^0, the distribution of the *reciprocals* of these price ratios could be considered. The counterparts to the asymmetric indices defined earlier by equations (16.54) and (16.55) are now $\sum_{i=1}^{n} s_i^0 (p_i^0 / p_i^1)$ and $\sum_{i=1}^{n} s_i^1 (p_i^0 / p_i^1)$, respectively. These are (stochastic) price indices going *backwards* from period 1 to 0. In order to make these indices comparable with other previous forward-looking indices, take the reciprocals of these indices (which leads to harmonic averages) and the following two indices are obtained:

$$P_{HL}(p^0, p^1, q^0, q^1) \equiv \frac{1}{\sum_{i=1}^{n} s_i^0 \dfrac{p_i^0}{p_i^1}} = \frac{1}{\sum_{i=1}^{n} s_i^0 \left(\dfrac{p_i^1}{p_i^0}\right)^{-1}} \qquad (16.59)$$

$$\begin{aligned} P_{HP}(p^0, p^1, q^0, q^1) &\equiv \frac{1}{\sum_{i=1}^{n} s_i^1 \dfrac{p_i^0}{p_i^1}} = \frac{1}{\sum_{i=1}^{n} s_i^1 \left(\dfrac{p_i^1}{p_i^0}\right)^{-1}} \\ &= P_P(p^0, p^1, q^0, q^1) \end{aligned} \qquad (16.60)$$

using equation (15.9) in Chapter 15. Thus the reciprocal stochastic price index defined by equation (16.60) turns out to equal the fixed basket Paasche price index, P_P. This stochastic approach is the natural one for studying sampling problems associated with implementing a Paasche price index. The other asymmetrically weighted reciprocal stochastic price index defined by the formula (16.59) has no author's name associated with it but it was noted by

[59] For a proof of this assertion, see Balk and Diewert (2001).

[60] In fact, this index suffers from the same upward bias as the Carli index in that $P_u(p^0,p^1,q^0,q^1)P_u(p^1,p^0,q^1,q^0) \geq 1$. To prove this, note that the previous inequality is equivalent to $[P_u(p^1,p^0,q^1,q^0)]^{-1} \leq P_u(p^0,p^1,q^0,q^1)$ and this inequality follows from the fact that a weighted harmonic mean of n positive numbers is equal or less than the corresponding weighted arithmetic mean; see Hardy, Littlewood and Pólya (1934, p. 26).

Fisher (1922, p. 467) as his index number formula 13. Vartia (1978, p. 272) called this index *the harmonic Laspeyres index* and his terminology will be used.

16.89 Now consider the class of symmetrically weighted reciprocal price indices defined as:

$$P_{mr}(p^0, p^1, q^0, q^1) \equiv \frac{1}{\sum_{i=1}^{n} m(s_i^0, s_i^1)\left(\frac{p_i^1}{p_i^0}\right)^{-1}} \qquad (16.61)$$

where, as usual, $m(s_i^0,s_i^1)$ is a homogeneous symmetric mean of the period 0 and 1 expenditure shares on commodity i. However, none of the indices defined by equations (16.59) to (16.61) satisfies the time reversal test.

16.90 The fact that Theil's index number formula P_T satisfies the time reversal test leads to a preference for Theil's index as the "best" weighted stochastic approach.

16.91 The main features of the weighted stochastic approach to index number theory can be summarized as follows. It is first necessary to pick two periods and a transactions domain of definition. As usual, each value transaction for each of the n commodities in the domain of definition is split up into price and quantity components. Then, assuming there are no new commodities or no disappearing commodities, there are n price relatives p_i^1/p_i^0 pertaining to the two situations under consideration along with the corresponding $2n$ expenditure shares. The weighted stochastic approach just assumes that these n relative prices, or some transformation of these price relatives, $f(p_i^1/p_i^0)$, have a discrete statistical distribution, where the ith probability, $\rho_i = m(s_i^0,s_i^1)$, is a function of the expenditure shares pertaining to commodity i in the two situations under consideration, s_i^0 and s_i^1. Different price indices result, depending on how the functions f and m are chosen. In Theil's approach, the transformation function f is the natural logarithm and the mean function m is the simple unweighted arithmetic mean.

16.92 There is a third aspect to the weighted stochastic approach to index number theory: it has to be decided what *single number* best summarizes the distribution of the n (possibly transformed) price relatives. In the above analysis, the *mean* of the discrete distribution was chosen as the "best" summary measure for the distribution of the (possibly transformed) price relatives; but other measures are possible. In particular, the *weighted median* or various *trimmed means* are often suggested as the "best" measure of central tendency because these measures minimize the influence of outliers. Detailed discussion of these alternative measures of central tendency is, however, beyond the scope of this chapter. Additional material on stochastic approaches to index number theory and references to the literature can be found in Clements and Izan (1981; 1987), Selvanathan and Rao (1994), Diewert (1995b), Cecchetti (1997) and Wynne (1997; 1999).

16.93 Instead of taking the above stochastic approach to index number theory, it is possible to take the same raw data that is used in this approach but use an axiomatic approach. Thus, in the following section, the price index is regarded as a value-weighted function of the n price relatives and the test approach to index number theory is used in order to determine the functional form for the price index. Put another way, the axiomatic approach in the next section looks at the *properties* of alternative descriptive statistics that aggregate the individual price relatives (weighted by their economic importance) into summary measures of price change in an attempt to find the "best" summary measure of price change. Thus the axiomatic approach pursued below can be viewed as a branch of the theory of descriptive statistics.

The second axiomatic approach to bilateral price indices

The basic framework and some preliminary tests

16.94 As mentioned in paragraphs 16.1 to 16.10, one of Walsh's approaches to index number theory was an attempt to determine the "best" weighted average of the price relatives, r_i.[61] This is equivalent to using an axiomatic approach to try to determine the "best" index of the form $P(r,v^0,v^1)$, where v^0 and v^1 are the vectors of expenditures on the n commodities

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[61] Fisher also took this point of view when describing his approach to index number theory:

> An index number of the prices of a number of commodities is an average of their price relatives. This definition has, for concreteness, been expressed in terms of prices. But in like manner, an index number can be calculated for wages, for quantities of goods imported or exported, and, in fact, for any subject matter involving divergent changes of a group of magnitudes. Again, this definition has been expressed in terms of time. But an index number can be applied with equal propriety to comparisons between two places or, in fact, to comparisons between the magnitudes of a group of elements under any one set of circumstances and their magnitudes under another set of circumstances (Fisher (1922, p. 3)).

In setting up his axiomatic approach, Fisher imposed axioms on the price and quantity indices written as functions of the two price vectors, p^0 and p^1, and the two quantity vectors, q^0 and q^1; i.e., he did not write his price index in the form $P(r,v^0,v^1)$ and impose axioms on indices of this type. Of course, in the end, his ideal price index turned out to be the geometric mean of the Laspeyres and Paasche price indices and, as was seen in Chapter 15, each of these indices can be written as expenditure share weighted averages of the n price relatives, $r_i \equiv p_i^1/p_i^0$.

during periods 0 and 1.[62] Initially, rather than starting with indices of the form $P(r,v^0,v^1)$, indices of the form $P(p^0,p^1,v^0,v^1)$ will be considered, since this framework will be more comparable to the first bilateral axiomatic framework taken in paragraphs 16.30 to 16.73. As will be seen below, if the invariance to changes in the units of measurement test is imposed on an index of the form $P(p^0,p^1,v^0,v^1)$, then $P(p^0,p^1,v^0,v^1)$ can be written in the form $P(r,v^0,v^1)$.

16.95 Recall that the product test (16.17) was used to define the quantity index $Q(p^0,p^1,q^0,q^1) \equiv V^1/V^0P(p^0,p^1,q^0,q^1)$ that corresponded to the bilateral price index $P(p^0,p^1,q^0,q^1)$. A similar product test holds in the present framework; i.e., given that the functional form for the price index $P(p^0,p^1,v^0,v^1)$ has been determined, then the corresponding *implicit quantity index* can be defined in terms of P as follows:

$$Q(p^0,p^1,v^0,v^1) \equiv \frac{\sum_{i=1}^{n} v_i^1}{\left(\sum_{i=1}^{n} v_i^0\right) P(p^0,p^1,v^0,v^1)} \qquad (16.62)$$

16.96 In paragraphs 16.30 to 16.73, the price and quantity indices $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ were determined *jointly*; i.e., not only were axioms imposed on $P(p^0,p^1,q^0,q^1)$, but they were also imposed on $Q(p^0,p^1,q^0,q^1)$, and the product test (16.17) was used to translate these tests on Q into tests on P. In this section, this approach will not be followed: only tests on $P(p^0,p^1,v^0,v^1)$ will be used in order to determine the "best" price index of this form. Thus, there is a parallel theory for quantity indices of the form $Q(q^0,q^1,v^0,v^1)$, where it is attempted to find the "best" value weighted average of the quantity relatives, q_i^1/q_i^0.[63]

16.97 For the most part, the tests which will be imposed on the price index $P(p^0,p^1,v^0,v^1)$ in this section are counterparts to the tests that were imposed on the price index $P(p^0,p^1,q^0,q^1)$ in paragraphs 16.30 to 16.73. It will be assumed that every component of each price and value

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[62] Chapter 3 in Vartia (1976) considered a variant of this axiomatic approach.

[63] It turns out that the price index that corresponds to this "best" quantity index, defined as $P^*(q^0,q^1,v^0,v^1) \equiv \sum_{i=1}^{n} v_i^1 \Big/ \left[\sum_{i=1}^{n} v_i^0 Q(q^0,q^1,v^0,v^1)\right]$, will not equal the "best" price index, $P(p^0,p^1,v^0,v^1)$. Thus the axiomatic approach used here generates *separate* "best" price and quantity indices whose product does not equal the value ratio in general. This is a disadvantage of the second axiomatic approach to bilateral indices compared to the first approach studied above.

vector is positive; i.e., $p^t >> 0_n$ and $v^t >> 0_n$ for $t = 0,1$. If it is desired to set $v^0 = v^1$, the common expenditure vector is denoted by v; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

16.98 The first two tests are straightforward counterparts to the corresponding tests in paragraph 16.34.

T1: *Positivity*: $P(p^0,p^1,v^0,v^1) > 0$

T2: *Continuity*: $P(p^0,p^1,v^0,v^1)$ is a continuous function of its arguments

T3: *Identity or constant prices test*: $P(p,p,v^0,v^1) = 1$

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the value vectors are. Note that the two value vectors are allowed to be different in the above test.

Homogeneity tests

16.99 The following four tests restrict the behaviour of the price index P as the scale of any one of the four vectors p^0,p^1,v^0,v^1 changes.

T4: *Proportionality in current prices*

:$P(p^0,\lambda p^1,v^0,v^1) = \lambda P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. This test is the counterpart to test T5 in paragraph 16.37.

16.100 In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T5: *Inverse proportionality in base period prices*:

$P(\lambda p^0,p^1,v^0,v^1) = \lambda^{-1}P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0. This test is the counterpart to test T6 in paragraph 16.39.

16.101 The following two homogeneity tests can also be regarded as invariance tests.

T6: *Invariance to proportional changes in current period values*:

$P(p^0,p^1,v^0,\lambda v^1) = P(p^0,p^1,v^0,v^1)$ for all $\lambda > 0$

That is, if current period values are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 1 value vector v^1.

T7: *Invariance to proportional changes in base period values*:

$P(p^0,p^1,\lambda v^0,v^1) = P(p^0,p^1,v^0,v^1)$ for all $\lambda > 0$

That is, if base period values are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 0 value vector v^0.

16.102 T6 and T7 together impose the property that the price index P does not depend on the *absolute* magnitudes of the value vectors v^0 and v^1. Using test T6 with $\lambda = 1/\sum_{i=1}^{n} v_i^1$ and using test T7 with $\lambda = 1/\sum_{i=1}^{n} v_i^0$, it can be seen that P has the following property:

$$P(p^0, p^1, v^0, v^1) = P(p^0, p^1, s^0, s^1) \qquad (16.63)$$

where s^0 and s^1 are the vectors of expenditure shares for periods 0 and 1; i.e., the *i*th component of s^t is $s_i^t \equiv v_i^t / \sum_{k=1}^{n} v_k^t$ for $t = 0,1$. Thus the tests T6 and T7 imply that the price index function P is a function of the two price vectors p^0 and p^1 and the two vectors of expenditure shares, s^0 and s^1.

16.103 Walsh (1901, p. 104) suggested the spirit of tests T6 and T7 as the following quotation indicates: "What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [i.e., the price relatives] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered."

16.104 Walsh also realized that weighting the *i*th price relative r_i by the arithmetic mean of the value weights in the two periods under consideration, $(1/2)[v_i^0 + v_i^1]$ would give too much weight to the expenditures of the period that had the highest level of prices:

> At first sight it might be thought sufficient to add up the weights of every class at the two periods and to divide by two. This would give the (arithmetic) mean size of every class over the two periods together. But such an operation is manifestly wrong. In the first place, the sizes of the classes at each period are reckoned in the money of the period, and if it happens that the exchange value of money has fallen, or prices in general have risen, greater influence upon the result would be given to the weighting of the second period; or if prices in general have fallen, greater influence would be given to the weighting of the first period. Or in a comparison between two countries, greater influence would be given to the weighting of the country with the higher level of prices. But it is plain that *the one period, or the one country, is as important, in our comparison between them, as the other, and the weighting in the averaging of their weights should really be even* (Walsh (1901, pp. 104-105)).

16.105 As a solution to the above weighting problem, Walsh (1901, p. 202; 1921a, p. 97) proposed the following *geometric price index*:

$$P_{GW}(p^0, p^1, v^0, v^1) \equiv \prod_{i=1}^{n} \left(\frac{p_i^1}{p_i^0} \right)^{w_i} \qquad (16.64)$$

where the *i*th weight in the above formula was defined as

$$w_i \equiv \frac{(v_i^0 v_i^1)^{1/2}}{\sum_{k=1}^{n} (v_k^0 v_k^1)^{1/2}} = \frac{(s_i^0 s_i^1)^{1/2}}{\sum_{k=1}^{n} (s_k^0 s_k^1)^{1/2}} \qquad i = 1,...,n. \qquad (16.65)$$

The second equation in (16.65) shows that Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ can also be written as a function of the expenditure share vectors, s^0 and s^1; i.e., $P_{GW}(p^0,p^1,v^0,v^1)$ is homogeneous of degree zero in the components of the value vectors v^0 and v^1 and so $P_{GW}(p^0,p^1,v^0,v^1) = P_{GW}(p^0,p^1,s^0,s^1)$. Walsh came very close to deriving the Törnqvist–Theil index defined earlier by equation (16.48).[64]

Invariance and symmetry tests

16.106 The next five tests are *invariance* or *symmetry tests* and four of them are direct counterparts to similar tests in paragraphs 16.42 to 16.46 above. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed.

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[64] Walsh's index could be derived using the same arguments as Theil, except that the geometric average of the expenditure shares $(s_i^0 s_i^1)^{1/2}$ could be taken as a preliminary probability weight for the *i*th logarithmic price relative, ln r_i. These preliminary weights are then normalized to add up to unity by dividing by their sum. It is evident that Walsh's geometric price index will closely approximate Theil's index using normal time series data. More formally, regarding both indices as functions of p^0,p^1,v^0,v^1, it can be shown that $P_W(p^0,p^1,v^0,v^1)$ approximates $P_T(p^0,p^1,v^0,v^1)$ to the second order around an equal price (i.e., $p^0 = p^1$) and quantity (i.e., $q^0 = q^1$) point.

T8: *Commodity reversal test* (or invariance to changes in the ordering of commodities):

$P(p^0*,p^1*,v^0*,v^1*) = P(p^0,p^1,v^0,v^1)$

where p^t* denotes a permutation of the components of the vector p^t and v^t* denotes the same permutation of the components of v^t for $t = 0,1$. The term "commodity reversal test" is due to Fisher (1922; p. 63) but Walter Lane suggested a more appropriate name for the test might be the "commodity permutation test".

16.107 The next test asks that the index be invariant to changes in the units of measurement.

T9: *Invariance to changes in the units of measurement* (commensurability test):

$P(\alpha_1 p_1^0,...,\alpha_n p_n^0; \alpha_1 p_1^1,...,\alpha_n p_n^1; v_1^0,...,v_n^0; v_1^1,...,v_n^1) =$

$P(p_1^0,...,p_n^0; p_1^1,...,p_n^1; v_1^0,...,v_n^0; v_1^1,...,v_n^1)$ for all $\alpha_1 > 0, \ldots, \alpha_n > 0$

That is, the price index does not change if the units of measurement for each commodity are changed. Note that the expenditure on commodity i during period t, v_i^t, does not change if the unit by which commodity i is measured changes.

16.108 The last test has a very important implication. Let $\alpha_1 = 1/p_1^0, \ldots, \alpha_n = 1/p_n^0$ and substitute these values for the α_i into the definition of the test. The following equation is obtained:

$$P(p^0, p^1, v^0, v^1) = P(1_n, r, v^0, v^1) \equiv P^*(r, v^0, v^1) \qquad (16.66)$$

where 1_n is a vector of ones of dimension n and r is a vector of the price relatives; i.e., the ith component of r is $r_i \equiv p_i^1/p_i^0$. Thus, if the commensurability test T9 is satisfied, then the price index $P(p^0,p^1,v^0,v^1)$, which is a function of $4n$ variables, can be written as a function of $3n$ variables, $P^*(r, v^0,v^1)$, where r is the vector of price relatives and $P^*(r, v^0,v^1)$ is defined as $P(1_n,r,v^0,v^1)$.

16.109 The next test asks that the formula be invariant to the period chosen as the base period.

T10: *Time reversal test*: $P(p^0,p^1,v^0,v^1) = 1/P(p^1,p^0,v^1,v^0)$

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one commodity case when the price index is simply the single price ratio, this test will be satisfied (as are all the other tests listed in this section).

16.110 The next test is a variant of the circularity test, introduced in paragraphs 15.76 to 15.97 of Chapter 15.[65]

T11: *Transitivity in prices for fixed value weights*:

$P(p^0,p^2,v^r,v^s) = P(p^0,p^1,v^r,v^s)P(p^1,p^2,v^r,v^s)$

In this test, the expenditure weighting vectors, v^r and v^s, are held constant while making all price comparisons. Given that these weights are held constant, however, the test asks that the product of the index going from period 0 to 1, $P(p^0,p^1,v^r,v^s)$, times the index going from period 1 to 2, $P(p^1,p^2,v^r,v^s)$, should equal the direct index that compares the prices of period 2 with those of period 0, $P(p^0,p^2,v^r,v^s)$. This test is a many-commodity counterpart to a property that obviously holds for a single price relative.

16.111 The final test in this section captures the idea that the value weights should enter the index number formula in a symmetric manner.

T12: *Value weights symmetry test*:

$P(p^0,p^1,v^0,v^1) = P(p^0,p^1,v^1,v^0)$

That is, if the expenditure vectors for the two periods are interchanged, then the price index remains invariant. This property means that, if values are used to weight the prices in the index number formula, then the period 0 values v^0 and the period 1 values v^1 must enter the formula in a symmetric or even-handed manner.

A mean value test

16.112 The next test is a *mean value test*.

T13: *Mean value test for prices*:

$$\min_i (p_i^1/p_i^0 : i=1,...,n) \le P(p^0, p^1, v^0, v^1) \le \max_i (p_i^1/p_i^0 : i=1,...,n) \qquad (16.67)$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is to be interpreted as an average of the n price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

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[65] See equation (15.77) in Chapter 15.

Monotonicity tests

16.113 The next two tests in this section are *monotonicity tests*; i.e., how should the price index $P(p^0,p^1,v^0,v^1)$ change as any component of the two price vectors p^0 and p^1 increases.

T14: *Monotonicity in current prices*:

$P(p^0,p^1,v^0,v^1) < P(p^0,p^2,v^0,v^1)$ if $p^1 < p^2$

That is, if some period 1 price increases and none decrease, then the price index must increase (holding the value vectors fixed), so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1 for fixed p^0, v^0 and v^1.

T15: *Monotonicity in base prices*:

$P(p^0,p^1,v^0,v^1) > P(p^2,p^1,v^0,v^1)$ if $p^0 < p^2$

That is, if any period 0 price increases and none decrease, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0 for fixed p^1, v^0 and v^1.

Weighting tests

16.114 The above tests are not sufficient to determine the functional form of the price index; for example, it can be shown that both Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by equation (16.65) and the Törnqvist–Theil index $P_T(p^0,p^1,v^0,v^1)$ defined by equation (16.48) satisfy all of the above axioms. Thus, at least one more test will be required in order to determine the functional form for the price index $P(p^0,p^1,v^0,v^1)$.

16.115 The tests proposed thus far do not specify exactly how the expenditure share vectors s^0 and s^1 are to be used in order to weight, say, the first price relative, $p_1{}^1/p_1{}^0$. The next test says that only the expenditure shares $s_1{}^0$ and $s_1{}^1$ pertaining to the first commodity are to be used in order to weight the prices that correspond to commodity 1, $p_1{}^1$ and $p_1{}^0$.

T16. *Own share price weighting*:

$$P(p_1^0,1,...,1\,;\,p_1^1,1,...,1\,;\,v^0,v^1) = f\left(p_1^0,p_1^1,\left[v_1^0\Big/\sum_{k=1}^{n}v_k^0\right],\left[v_1^1\Big/\sum_{k=1}^{n}v_k^1\right]\right) \quad (16.68)$$

Note that $v_1^t/\sum_{k=1}^{n}v_k^t$ equals $s_1{}^t$, the expenditure share for commodity 1 in period t. The above test says that if all the prices are set equal to 1 except the prices for commodity 1 in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity 1 and the two expenditure shares for commodity 1. The axiom says that a function of 2 + 2n variables is actually only a function of four variables.[66]

16.116 Of course, if test T16 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property:

$$P(1,...,1, p_i^0, 1,...,1 \,;\, 1,...,1, p_i^1, 1,...,1 \,;\, v^0 \,;\, v^1) = f\left(p_i^0, p_i^1, \left[v_i^0 \Big/ \sum_{k=1}^{n} v_k^0\right], \left[v_i^1 \Big/ \sum_{k=1}^{n} v_k^1\right]\right) \qquad i = 1,...,n. \tag{16.69}$$

Equation (16.69) says that, if all the prices are set equal to 1 except the prices for commodity i in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity i and the two expenditure shares for commodity i.

16.117 The final test that also involves the weighting of prices is the following one:

T17: *Irrelevance of price change with tiny value weights*:

$$P(p_1^0, 1,...,1 \,;\, p_1^1, 1,...,1 \,;\, 0, v_2^0,...,v_n^0 \,;\, 0, v_2^1,...,v_n^1) = 1 \tag{16.70}$$

The test T17 says that, if all the prices are set equal to 1 except the prices for commodity 1 in the two periods, and the expenditures on commodity 1 are zero in the two periods but the expenditures on the other commodities are arbitrarily given, then the index is equal to 1.[67] Thus, roughly speaking, if the value weights for commodity 1 are tiny, then it does not matter what the price of commodity 1 is during the two periods.

16.118 Of course, if test T17 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property: for i = 1,…,n:

$$P(1,...,1, p_i^0, 1,...,1 \,;\, 1,...,1, p_i^1, 1,...,1 \,;\, v_1^0,...,0,...,v_n^0 \,;\, v_1^1,...,0,...,v_n^1) = 1 \tag{16.71}$$

Equation (16.71) says that, if all the prices are set equal to 1 except the prices for commodity i in the two periods, and the expenditures on commodity i are 0 during the two periods but the other expenditures in the two periods are arbitrarily given, then the index is equal to 1.

[66] In the economics literature, axioms of this type are known as separability axioms.

[67] Strictly speaking, since all prices and values are required to be positive, the left-hand side of equation (16.70) should be replaced by the limit as the commodity 1 values, v_1^0 and v_1^1, approach 0.

16.119 This completes the listing of tests for the approach to bilateral index number theory based on the weighted average of price relatives. It turns out that the above tests are sufficient to imply a specific functional form for the price index, as seen in the next section.

The Törnqvist–Theil price index and the second test approach to bilateral indices

16.120 In Appendix 16.1 to this chapter, it is shown that, if the number of commodities n exceeds two and the bilateral price index function $P(p^0,p^1,v^0,v^1)$ satisfies the 17 axioms listed above, then P must be the Törnqvist–Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined by equation (16.48).[68] Thus the 17 properties or tests listed in paragraphs 16.94 to 16.129 provide an axiomatic characterization of the Törnqvist–Theil price index, just as the 20 tests listed in paragraphs 16.30 to 16.73 provided an axiomatic characterization of the Fisher ideal price index.

16.121 Obviously, there is a parallel axiomatic theory for quantity indices of the form $Q(q^0,q^1,v^0,v^1)$ that depend on the two quantity vectors for periods 0 and 1, q^0 and q^1, as well as on the corresponding two expenditure vectors, v^0 and v^1. Thus, if $Q(q^0,q^1,v^0,v^1)$ satisfies the quantity counterparts to tests T1 to T17, then Q must be equal to the Törnqvist–Theil quantity index $Q_T(q^0,q^1,v^0,v^1)$ defined, as follows:

$$\ln Q_T(q^0,q^1,v^0,v^1) \equiv \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1) \ln\left(\frac{q_i^1}{q_i^0}\right) \qquad (16.72)$$

where as usual, the period t expenditure share on commodity i, s_i^t, is defined as $v_i^t / \sum_{k=1}^{n} v_k^t$ for $i = 1,\ldots,n$ and $t = 0,1$.

16.122 Unfortunately, the implicit Törnqvist–Theil price index, $P_{IT}(q^0,q^1,v^0,v^1)$ that corresponds to the Törnqvist–Theil quantity index Q_T, defined by equation (16.72) using the product test, is not equal to the direct Törnqvist–Theil price index $P_T(p^0,p^1,v^0,v^1)$, defined by

[68] The Törnqvist–Theil price index satisfies all 17 tests, but the proof in Appendix 16.1 does not use all these tests to establish the result in the opposite direction: tests 5, 13, 15 and one of 10 or 12 were not required in order to show that an index satisfying the remaining tests must be the Törnqvist–Theil price index. For alternative characterizations of the Törnqvist–Theil price index, see Balk and Diewert (2001) and Hillinger (2002).

equation (16.48). The product test equation that defines P_{IT} in the present context is given by the following equation:

$$P_{IT}(q^0,q^1,v^0,v^1) \equiv \frac{\sum_{i=1}^{n} v_i^1}{\left(\sum_{i=1}^{n} v_i^0\right) Q_T(q^0,q^1,v^0,v^1)} \qquad (16.73)$$

The fact that the direct Törnqvist–Theil price index P_T is not in general equal to the implicit Törnqvist–Theil price index P_{IT} ,defined by equation (16.73), is something of a disadvantage compared to the axiomatic approach outlined in paragraphs 16.30 to 16.73, which led to the Fisher ideal price and quantity indices being considered "best". Using the Fisher approach meant that it was not necessary to decide whether the aim was to find a "best" price index or a "best" quantity index: the theory outlined in paragraphs 16.30 to 16.73 determined both indices simultaneously. In the Törnqvist–Theil approach outlined in this section, however, it is necessary to choose between a "best" price index or a "best" quantity index.[69]

16.123 Other tests are of course possible. A counterpart to Test T16 in paragraph 16.49, the Paasche and Laspeyres bounding test, is the following *geometric Paasche and Laspeyres bounding test*:

$$\begin{aligned} &P_{GL}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GP}(p^0,p^1,v^0,v^1) \text{ or} \\ &P_{GP}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GL}(p^0,p^1,v^0,v^1) \end{aligned} \qquad (16.74)$$

where the logarithms of the geometric Laspeyres and geometric Paasche price indices, P_{GL} and P_{GP}, are defined as follows:

$$\ln P_{GL}(p^0,p^1,v^0,v^1) \equiv \sum_{i=1}^{n} s_i^0 \ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (16.75)$$

$$\ln P_{GP}(p^0,p^1,v^0,v^1) \equiv \sum_{i=1}^{n} s_i^1 \ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (16.76)$$

As usual, the period t expenditure share on commodity i, s_i^t, is defined as $v_1^t / \sum_{k=1}^{n} v_k^t$ for i = 1,…,n and t = 0,1. It can be shown that the Törnqvist–Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined by equation (16.48) satisfies this test, but the geometric Walsh price index

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[69] Hillinger (2002) suggested taking the geometric mean of the direct and implicit Törnqvist–Theil price indices in order to resolve this conflict. Unfortunately, the resulting index is not "best" for either set of axioms that were suggested in this section.

$P_{GW}(p^0,p^1,v^0,v^1)$ defined by equation (16.65) does not. The geometric Paasche and Laspeyres bounding test was not included as a primary test in this section because it was not known a priori what form of averaging of the price relatives (e. g., geometric or arithmetic or harmonic) would turn out to be appropriate in this test framework. The test (16.74) is an appropriate one if it has been decided that geometric averaging of the price relatives is the appropriate framework, since the geometric Paasche and Laspeyres indices correspond to "extreme" forms of value weighting in the context of geometric averaging and it is natural to require that the "best" price index lies between these extreme indices.

16.124 Walsh (1901, p. 408) pointed out a problem with his geometric price index defined by equation (16.65), which also applies to the Törnqvist–Theil price index, $P_T(p^0,p^1,v^0,v^1)$, defined by equation (16.48): these geometric type indices do not give the "right" answer when the quantity vectors are constant (or proportional) over the two periods. In this case, Walsh thought that the "right" answer must be the Lowe index, which is the ratio of the costs of purchasing the constant basket during the two periods. Put another way, the geometric indices P_{GW} and P_T do not satisfy the fixed basket test T4 in paragraph 16.35. What then was the argument that led Walsh to define his geometric average type index P_{GW}? It turns out that he was led to this type of index by considering another test, which will now be explained.

16.125 Walsh (1901, pp. 228-231) derived his test by considering the following very simple framework. Let there be only two commodities in the index and suppose that the expenditure share on each commodity is equal in each of the two periods under consideration. The price index under these conditions is equal to $P(p_1^0,p_2^0;p_1^1,p_2^1;v_1^0,v_2^0;v_1^1,v_2^1) = P^*(r_1,r_2;1/2,1/2;1/2,1/2) \equiv m(r_1,r_2)$, where $m(r_1,r_2)$ is a symmetric mean of the two price relatives, $r_1 \equiv p_1^1/p_1^0$ and $r_2 \equiv p_2^1/p_2^0$.[70] In this framework, Walsh then proposed the following *price relative reciprocal test*:

$$m(r_1, r_1^{-1}) = 1 \qquad (16.77)$$

Thus, if the value weighting for the two commodities is equal over the two periods and the second price relative is the reciprocal of the first price relative r_1, then Walsh (1901, p. 230) argued that the overall price index under these circumstances ought to equal one, since the relative fall in one price is exactly counterbalanced by a rise in the other and both

[70] Walsh considered only the cases where m was the arithmetic, geometric and harmonic means of r_1 and r_2.

commodities have the same expenditures in each period. He found that the geometric mean satisfied this test perfectly but the arithmetic mean led to index values greater than one (provided that r_1 was not equal to one) and the harmonic mean led to index values that were less than one, a situation which was not at all satisfactory.[71] Thus he was led to some form of geometric averaging of the price relatives in one of his approaches to index number theory.

16.126 A generalization of Walsh's result is easy to obtain. Suppose that the mean function, $m(r_1,r_2)$, satisfies Walsh's reciprocal test (16.77) and, in addition, m is a homogeneous mean, so that it satisfies the following property for all $r_1 > 0$, $r_2 > 0$ and $\lambda > 0$:

$$m(\lambda r_1, \lambda r_2) = \lambda m(r_1, r_2) \quad (16.78)$$

Let $r_1 > 0$, $r_2 > 0$. Then

$$\begin{aligned} m(r_1, r_2) &= \left(\frac{r_1}{r_1}\right) m(r_1, r_2) \\ &= r_1 m\left(\frac{r_1}{r_1}, \frac{r_2}{r_1}\right) \qquad \text{using (16.78) with } \lambda = \frac{1}{r_1} \quad (16.79) \\ &= r_1 m \geq \left(1, \frac{r_2}{r_1}\right) = r_1 f\left(\frac{r_2}{r_1}\right) \end{aligned}$$

where the function of one (positive) variable $f(z)$ is defined as

$$f(z) \equiv m(1, z) \quad (16.80)$$

Using equation (16.77):

$$\begin{aligned} 1 &= m(r_1, r_1^{-1}) \\ &= \left(\frac{r_1}{r_1}\right) m(r_1, r_1^{-1}) \quad (16.81) \\ &= r_1 m(1, r_1^{-2}) \qquad \text{using (16.78) with } \lambda = \frac{1}{r_1} \end{aligned}$$

Using equation (16.80), equation (16.81) can be rearranged in the following form:

$$f(r_1^{-2}) = r_1^{-1} \quad (16.82)$$

Letting $z \equiv r_1^{-2}$ so that $z^{1/2} = r_1^{-1}$, equation (16.82) becomes:

$$f(z) = z^{1/2} \quad (16.83)$$

[71] "This tendency of the arithmetic and harmonic solutions to run into the ground or to fly into the air by their excessive demands is clear indication of their falsity" (Walsh (1901, p. 231)).

Now substitute equation (16.83) into equation (16.79) and the functional form for the mean function $m(r_1,r_2)$ is determined:

$$m(r_1,r_2) = r_1 f\left(\frac{r_2}{r_1}\right) = r_1\left(\frac{r_2}{r_1}\right)^{1/2} = r_1^{1/2} r_2^{1/2} \qquad (16.84)$$

Thus, the geometric mean of the two price relatives is the only homogeneous mean that will satisfy Walsh's price relative reciprocal test.

16.127 There is one additional test that should be mentioned. Fisher (1911; p. 401) introduced this test in his first book that dealt with the test approach to index number theory. He called it the *test of determinateness as to prices* and described it as follows: "A price index should not be rendered zero, infinity, or indeterminate by an individual price becoming zero. Thus, if any commodity should in 1910 be a glut on the market, becoming a 'free good', that fact ought not to render the index number for 1910 zero." In the present context, this test could be interpreted as the following one: if any single price p_i^0 or p_i^1 tends to zero, then the price index $P(p^0,p^1,v^0,v^1)$ should not tend to zero or plus infinity. However, with this interpretation of the test, which regards the values v_i^t as remaining constant as the p_i^0 or p_i^1 tends to zero, none of the commonly used index number formulae would satisfy this test. Hence this test should be interpreted as a test that applies to price indices $P(p^0,p^1,q^0,q^1)$ of the type studied in paragraphs 16.30 to 16.73, which is how Fisher intended the test to apply. Thus, Fisher's price determinateness test should be interpreted as follows: if any single price p_i^0 or p_i^1 tends to zero, then the price index $P(p^0,p^1,q^0,q^1)$ should not tend to zero or plus infinity. With this interpretation of the test, it can be verified that Laspeyres, Paasche and Fisher indices satisfy this test but the Törnqvist–Theil price index does not. Thus, when using the Törnqvist–Theil price index, care must be taken to bound the prices away from zero in order to avoid a meaningless index number value.

16.128 Walsh was aware that geometric average type indices such as the Törnqvist-Theil price index P_T or Walsh's geometric price index P_{GW} defined by equation (16.64) become somewhat unstable[72] as individual price relatives become very large or small:

[72] That is, the index may approach zero or plus infinity.

> Hence in practice the geometric average is not likely to depart much from the truth. Still, we have seen that when the classes [i.e., expenditures] are very unequal and the price variations are very great, this average may deflect considerably (Walsh (1901, p. 373)).
> In the cases of moderate inequality in the sizes of the classes and of excessive variation in one of the prices, there seems to be a tendency on the part of the geometric method to deviate by itself, becoming untrustworthy, while the other two methods keep fairly close together (Walsh (1901, p. 404)).

16.129 Weighing all the arguments and tests presented above, it seems that there may be a slight preference for the use of the Fisher ideal price index as a suitable target index for a statistical agency, but, of course, opinions may differ on which set of axioms is the most appropriate to use in practice.

The test properties of the Lowe and Young indices

16.130 The Young and Lowe indices were defined in Chapter 15. In the present section, the axiomatic properties of these indices with respect to their price arguments are developed.[73]

16.131 Let $q^b \equiv [q_1{}^b,...,q_n{}^b]$ and $p^b \equiv [p_1{}^b,...,p_n{}^b]$ denote the quantity and price vectors pertaining to some base year. The corresponding *base year expenditure shares* can be defined in the usual way as

$$s_i^b \equiv \frac{p_i^b q_i^b}{\sum_{k=1}^{n} p_k^b q_k^b} \qquad i = 1,...,n \qquad (16.85)$$

Let $s^b \equiv [s_1{}^b,...,s_n{}^b]$ denote the vector of base year expenditure shares. The Young (1812) price index between periods 0 and t is defined as follows:

$$P_Y(p^0, p^t, s^b) \equiv \sum_{i=1}^{n} s_i^b \left(\frac{p_i^t}{p_i^0} \right) \qquad (16.86)$$

The Lowe (1823, p. 316) price index[74] between periods 0 and t is defined as follows:

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[73] Baldwin (1990, p. 255) worked out a few of the axiomatic properties of the Lowe index.

[74] This index number formula is also precisely Bean and Stine's (1924, p. 31) Type A index number formula. Walsh (1901, p. 539) initially mistakenly attributed Lowe's formula to G. Poulett Scrope (1833), who wrote *Principles of political economy* in 1833 and suggested Lowe's formula without acknowledging Lowe's priority. But in his discussion of Fisher's (1921) paper, Walsh (1921b, p. 543-544) corrects his mistake on assigning Lowe's formula:

> What index number should you then use? It should be this: $\sum q\, p_1 / \sum q\, p_0$. This is the method used by Lowe within a year or two of one hundred years ago. In my [1901] book, I called it Scope's index number; but it should be called Lowe's. Note that in it are used quantities neither of a base year nor of a subsequent year. The quantities used should be rough estimates of what the quantities were throughout the period or epoch.

$$P_{Lo}(p^0, p^t, q^b) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{k=1}^{n} p_k^0 q_k^b} = \frac{\sum_{i=1}^{n} s_i^b \left(\frac{p_i^t}{p_i^b}\right)}{\sum_{k=1}^{n} s_k^b \left(\frac{p_k^0}{p_k^b}\right)} \qquad (16.87)$$

16.132 Drawing on the axioms listed above in this chapter, 12 desirable axioms for price indices of the form $P(p^0,p^1)$ are listed below. The period 0 and t price vectors, p^0 and p^t, are presumed to have strictly positive components.

T1: *Positivity*: $P(p^0,p^t) > 0$ if all prices are positive

T2: *Continuity*: $P(p^0,p^t)$ is a continuous function of prices

T3: *Identity test*: $P(p,p) = 1$

T4: *Homogeneity test for period* t *prices*: $P(p^0,\lambda p^t) = \lambda P(p^0,p^t)$ for all $\lambda > 0$

T5: *Homogeneity test for period* 0 *prices*: $P(\lambda p^0,p^t) = \lambda^{-1}P(p^0,p^t)$ for all $\lambda > 0$

T6: *Commodity reversal test*: $P(p^0,p^t) = P(p^{0*},p^{t*})$ where p^{0*} and p^{t*} denote the same permutation[75] of the components of the price vectors p^0 and p^t

T7: *Invariance to changes in the units of measurement (commensurability test)*

T8: *Time reversal test*: $P(p^t,p^0) = 1/P(p^0,p^t)$

T9: *Circularity or transitivity test*: $P(p^0,p^2) = P(p^0,p^1)P(p^1,p^2)$

T10: *Mean value test*: $\min\{p_i^t/p_i^0 : i = 1,\ldots,n\} \le P(p^0,p^t) \le \max\{p_i^t/p_i^0 : i = 1,\ldots,n\}$

T11: *Monotonicity test with respect to period* t *prices*: $P(p^0,p^t) < P(p^0,p^{t*})$ if $p^t < p^{t*}$

T12: *Monotonicity test with respect to period* 0 *prices*: $P(p^0,p^t) > P(p^{0*},p^t)$ if $p^0 < p^{0*}$

16.133 It is straightforward to show that the Lowe index defined by equation (16.87) satisfies all 12 of the axioms or tests listed above. Hence the Lowe index has very good axiomatic properties with respect to its price variables.[76]

16.134 It is straightforward to show that the Young index defined by equation (16.86) satisfies 10 of the 12 axioms, failing the time reversal test T8 and the circularity test T9. Thus

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[75] In applying this test to the Lowe and Young indices, it is assumed that the base year quantity vector q^b and the base year share vector s^b are subject to the same permutation.

[76] From the discussion in Chapter 15, it will be recalled that the main problem with the Lowe index occurs if the quantity weight vector q^b is not representative of the quantities that were purchased during the time interval between periods 0 and 1.

the axiomatic properties of the Young index are definitely inferior to those of the Lowe index.

16.135 In place of the Young index P_Y defined by (16.86), it possible to define the Geometric Young index (or the weighted Jevons index) as follows:

$$P_{GY}(p^0,p^t,s^b) \equiv \prod_{i=1}^{n}\left[\frac{p_i^t}{p_i^0}\right]^{s_i^b} . \qquad (16.88)$$

This index satisfies all 12 tests so it is as good as the Lowe index with respect to its axiomatic properties.

Appendix 16.1 Proof of the optimality of the Törnqvist–Theil price index in the second bilateral test approach

The tests (T1, T2, etc.) mentioned in this appendix are those presented in paragraphs 16.98 to 16.119.

1. Define $r_i \equiv p_i^1/p_i^0$ for $i = 1,\ldots,n$. Using T1, T9 and equation (16.66), $P(p^0,p^1,v^0,v^1) = P^*(r, v^0,v^1)$. Using T6, T7 and equation (16.63):

$$P(p^0, p^1, v^0, v^1) = P^*(r, s^0, s^1) \qquad \text{(A16.1.1)}$$

where s^t is the period t expenditure share vector for $t = 0,1$.

2. Let $x \equiv (x_1,\ldots,x_n)$ and $y \equiv (y_1,\ldots,y_n)$ be strictly positive vectors. The transitivity test T11 and equation (A16.1.1) imply that the function P^* has the following property:

$$P^*(x; s^0, s^1)P^*(y; s^0, s^1) = P^*(x_1y_1,\ldots,x_ny_n; s^0, s^1) \qquad \text{(A16.1.2)}$$

3. Using test T1, $P^*(r,s^0,s^1) > 0$ and using test T14, $P^*(r, s^0,s^1)$ is strictly increasing in the components of r. The identity test T3 implies that

$$P^*(1_n, s^0, s^1) = 1 \qquad \text{(A16.1.3)}$$

where 1_n is a vector of ones of dimension n. Using a result attributable to Eichhorn (1978, p. 66), it can be seen that these properties of P^* are sufficient to imply that there exist positive functions $\alpha_i(s^0,s^1)$ for $i = 1,\ldots,n$ such that P^* has the following representation:

$$\ln P^*(r, s^0, s^1) = \sum_{i=1}^{n}\alpha_i(s^0, s^1)\ln r_i \qquad \text{(A16.1.4)}$$

4. The continuity test T2 implies that the positive functions $\alpha_i(s^0,s^1)$ are continuous. For $\lambda > 0$, the linear homogeneity test T4 implies that

$$\ln P^*(\lambda r, s^0, s^1) = \ln \lambda P(r, s^0, s^1)$$
$$= \ln \lambda + \ln P^*(r, s^0, s^1).$$

$$Also, \ln P^*(\lambda r, s^0, s^1) = \sum_{i=1}^{n} \alpha_i (s^0, s^1) \ln \lambda r_i \qquad \text{using (A16.1.4)} \qquad \text{(A16.1.5)}$$
$$= \sum_{i=1}^{n} \alpha_i (s^0, s^1) \ln r_i + \sum_{i=1}^{n} \alpha_i (s^0, s^1) \ln \lambda$$
$$= \ln P^*(r, s^0, s^1) + \sum_{i=1}^{n} \alpha_i (s^0, s^1) \ln \lambda \qquad .$$

Therefore ln λ + ln P*(r,s^0,s^1) = ln P*(r,s^0,s^1) + $\sum_{i=1}^{n} \alpha_i (s^0, s^1) \ln \lambda$

Consequently, ln λ = ln λ $\sum_{i=1}^{n} \alpha_i (s^0, s^1)$

So $\sum_{i=1}^{n} \alpha_i (s^0, s^1) = 1$. (A16.1.6)

5. Using the weighting test T16 and the commodity reversal test T8, equations (16.69) hold. Equation (16.69) combined with the commensurability test T9 implies that *P** satisfies the following equation:

$$P^*(1,...,1, r_i, 1,...,1; s^0, s^1) = f(1, r_i, s_i^0, s_i^1); \qquad i = 1,...,n \qquad \text{(A16.1.7)}$$

for all $r_i > 0$ where *f* is the function defined in test T16.

6. Substitute equation (A16.1.7) into equation (A16.1.4) in order to obtain the following system of equations:

$$\ln f(1, r_i, s^0, s^1) = \ln P^*(1,...,1, r_i, 1,...,1; s^0, s^1) = \sum_{j=1}^{n} \alpha_j (s^0, s^1) \ln r_j = \alpha_i (s^0, s^1) \ln r_i \qquad \text{(A16.1.8)}$$

(since r_j=1 for $j \neq i$). Then, because the right-hand side of eqn. (16.8) involves only the *ith* elements of the vectors s^0 and s^1, the same must be true of the left-hand side, i.e., ln f(1,r_i,s_i^0,s_i^1).

But equation (A16.1.8) implies that the positive continuous function of 2*n* variables $\alpha_i(s^0,s^1)$ is constant with respect to all of its arguments except s_i^0 and s_i^1 and this property holds for each *i*. Thus each $\alpha_i(s^0,s^1)$ can be replaced by the positive continuous function of two variables $\beta_i(s_i^0,s_i^1)$ for $i = 1,\ldots,n$.[77] Now replace the $\alpha_i(s^0,s^1)$ in equation (A16.1.4) by the $\beta_i(s_i^0,s_i^1)$ for $i = 1,\ldots,n$ and the following representation for P^* is obtained:

$$\ln P^*(r,s^0,s^1) = \sum_{i=1}^{n} \beta_i(s_i^0,s_i^1)\ln r_i . \qquad \text{(A16.1.9)}$$

7. Equation (A16.1.6) implies that the functions $\beta_i(s_i^0,s_i^1)$ also satisfy the following restrictions:

$$\sum_{i=1}^{n} s_i^0 = 1; \text{ and } \sum_{i=1}^{n} s_i^1 = 1 \text{ implies } \sum_{i=1}^{n} \beta_i(s_i^0,s_i^1) = 1 \qquad \text{(A16.1.10)}$$

8. Assume that the weighting test T17 holds and substitute equation (16.71) into equation (A16.1.9) in order to obtain the following equation:

$$\beta_i(0,0)\ln\left(\frac{p_i^1}{p_i^0}\right) = 0; \qquad i = 1,\ldots,n \qquad \text{(A16.1.11)}$$

Since the p_i^1 and p_i^0 can be arbitrary positive numbers, it can be seen that equation (A16.1.11) implies

$$\beta_i(0,0) = 0; \qquad i = 1,\ldots,n \qquad \text{(A16.1.12)}$$

9. Assume that the number of commodities n is equal to or greater than 3. Using equations (A16.1.10) and (A16.1.12), Theorem 2 in Aczél (1987, p. 8) can be applied and the following functional form for each of the $\beta_i(s_i^0,s_i^1)$ is obtained:

$$\beta_i(s_i^0,s_i^1) = \gamma s_i^0 + (1-\gamma)s_i^1; \qquad i = 1,\ldots,n \qquad \text{(A16.1.13)}$$

where γ is a positive number satisfying $0 < \gamma < 1$.

10. Finally, the time reversal test T10 *or* the quantity weights symmetry test T12 can be used to show that γ must equal ½. Substituting this value for γ back into equation (A16.1.13) and then substituting that equation back into equation (A16.1.9), the functional form for P^* and hence P is determined as

$$\ln P(p^0,p^1,v^0,v^1) = \ln P^*(r,s^0,s^1) = \sum_{i=1}^{n} \frac{1}{2}(s_i^0 + s_i^1)\ln\left(\frac{p_i^1}{p_i^0}\right). \qquad \text{(A16.1.14)}$$

[77] More explicitly, $\beta_1(s_1^0,s_1^1) \equiv \alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ and so on. That is, in defining $\beta_1(s_1^0,s_1^1)$, the function $\alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ is used where all components of the vectors s^0 and s^1 except the first are set equal to an arbitrary positive number such as 1.